A comparison theorem for integrated stochastic Volterra models with application to the modelling of Lagrangian intermittency in turbulence

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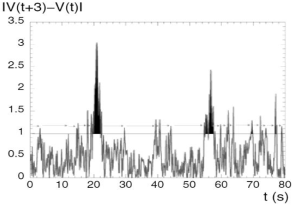
[1] This is an ongoing work, jointly with Mireille Bossy and Kerlyns Martínez.

Wind gusts are small-scale wind fluctuations that are by nature **intermittent**.



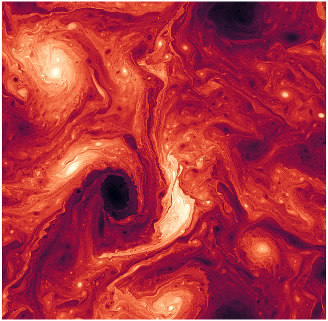
Given a time scale τ (here 3 seconds) and a threshold δ (here 1 m/s), characterising intermittent fluctuation $\|\Delta U(\tau)\| = \|U(t+\tau) - U(t)\| > \delta$ having **non Gaussian** properties

- Some predictive frameworks are ready to use, but assuming **Gaussian statistics**.
- Goal : develop a **stochastic model** that take into account **Kolomogorov's refined theory**. This involves stochastic processes with **memory**.
- Kolmogorov's theory predicts **multiscaling** such as anomalous power-laws emerging at the level of the velocity increments : $\mathbb{E}[|\Delta U(\tau)|^p] \simeq \tau^{\zeta(p)}$, with ζ **non-linear** function.

Plan of the talk

1. **Modelling with Volterra processes in turbulence**
   - Physical context: multifractality in turbulence
   - Effect of Volterra kernels in the statistics
2. **A weak comparison theorem for integrated Volterra processes**
   - The martingale approach
   - Path derivatives, functionnal Itô formula and PPDEs
   - Main result
3. **Applications**
   - Applications in view of the modelling
   - Application to the weak convergence of Markovian approximations

1. Modelling with Volterra processes in turbulence



A direct numerical simulation of 2D turbulence, provided by Nicolas Valade (Calisto Team INRIA)

- **Navier-Stokes** equation:
  $\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \frac{1}{\mathrm{Re}} \Delta \vec{u}$
- **Energy dissipation**:
  $\varepsilon(t,x) = \frac{\nu}{2} \langle \mathrm{trace} \nabla^T u \nabla u) \rangle (t,x)$
- In **Lagrangian** setting,

$$X_t = X_0 + \int_0^t u(s, X_s) ds$$
$$\varepsilon_t = \frac{\nu}{2} \langle \mathrm{trace} \nabla^T u \nabla u)(t, X_t) \rangle$$

(1) - Physical context: multifractality in turbulence

Kolmogorov's refined theory for fluctuations of the **energy dissipation** ε (can be seen as the volatility behind the velocity U): [Kolmogorov, 1962] [Frisch and Parisi, 1985] [Frisch, 1995]:

- **stationarity** and **scaling** : $\mathbb{E}[\varepsilon_t] = \nu\tau_\eta^{-2}$ (Kolmogorov 1941);
- **log-normality** of ε: with $\mathbb{V}\mathrm{ar}[\log \varepsilon_t] \simeq \log\left(\frac{\tau_L}{\tau_\eta}\right)$; $\qquad \tau_L = \frac{1}{\langle \|u\|^2\rangle(t)} \int_0^{+\infty} \langle u(t+\theta)u(t)\rangle d\theta$
- **multiscaling** of the one-point statistics: $\mathbb{E}[\varepsilon_t^p] \simeq \left(\frac{T_L}{\tau_\eta}\right)^{\zeta(p)}$, where $\zeta(p)$ is a non-linear convex function;
- **power-law scaling** for the **coarse-grained** dissipation and the velocity: in the inertial range, $\tau_\eta \ll \tau \ll T_L$,

$$\mathbb{E}\left[\left|\frac{1}{\tau}\int_t^{t+\tau} \varepsilon_s\right|^p\right] \simeq \tau^{\zeta(p)},$$

$$\mathbb{E}[|U(t+\tau) - U(t)|^p] \simeq \tau^{\zeta(p)}.$$

We construct a **stochastic model** for ε in the form $\varepsilon_t = \overline{\varepsilon}\exp(\gamma V_t - \frac{\gamma^2}{2}\operatorname{Var} V_t)$, where V is a Gaussian process to be determined, $\overline{\varepsilon} \in \mathbb{R}_+$, and $\gamma > 0$.

We find different proposals for the choice of V in the literature, in the form of stochastic **Volterra** processes:

- In [Forde et al., 2022], the authors consider the **fractional Brownian motion** of Riemann-Liouville $V_t = \int_0^t (t-s)^{-\frac{1}{2}+H} dW_s$. They demonstrate that for $\gamma \in (0, \sqrt{2})$, the measure $\xi_H(dt) = \exp(\gamma V_t - \frac{\gamma^2}{2}\operatorname{Var} V_t)dt$ is **locally multifractal** outside of zero in the limit $H \to 0$, i.e., for all $t \in (0,1)$,

$$\lim_{\tau \to 0} \frac{\log(\lim_{H\to 0} \mathbb{E}[\xi_H([t,t+\tau])^p])}{\log(\tau)} = \zeta(p) + p,$$

  with $\zeta(p) = -\frac{1}{2}\gamma^2(p^2 - p)$.

- [Letournel, 2022] proposes in his thesis to consider the **stationary** process $V_t = \int_{-\infty}^t \left[(t-r+\tau_\eta)^{H-\frac{1}{2}} - (t-r+\tau_L)^{H-\frac{1}{2}}\right] dW_r$, which is **well-defined for $H = 0$**, and in this case satisfies $\mathbb{E}[V_s V_t] \simeq \log_+ \frac{1}{t-s}$ for $\tau_\eta \ll s < t \ll \tau_L$, as well as $\mathbb{V}\text{ar}(V_t) \sim \log(\tau_L/\tau_\eta)$ in the limit $\tau_L/\tau_\eta \to +\infty$. However, **no rigorous proof** of multifractality is provided.

(1) - How to compare the effects of Volterra kernels ?

We aim to find a **stationary**, **locally multifractal** process that is **less expensive** to simulate than fBm as $H \to 0$.

- To achieve this, we will demonstrate a **comparison** result for integrated EVS models. Let $T > t_0 > 0$. For b measurable, $K, \overline{K} \in L^2([0,T],\mathbb{R})$, and $\phi : \mathbb{R} \to \mathbb{R}$ sufficiently regular, we define

$$X_T = \int_{t_0}^{T} b(s, V_s)ds;\ V_s = \int_0^s K(s-r)dW_r$$
$$\overline{X}_T = \int_{t_0}^{T} b(s, \overline{V}_s)ds;\ \overline{V}_s = \int_0^s \overline{K}(s-r)dW_r,$$

  and we are interested in the "weak" error $\mathscr{E}_T = |\mathbb{E}[\phi(X_T)] - \mathbb{E}[\phi(\overline{X}_T)]|$.

- Note that for $b(s,x) = \exp(\gamma x - \frac{\gamma^2}{2}\operatorname{Var} V_s)$, $\phi(x) = x^p$, $t_0 = t$, $T = t + \tau$, $K(s-r) = (s-r)^{H-\frac{1}{2}}$, we have

$$\mathbb{E}[\phi(X_T)] = \mathbb{E}[\xi_H([t,t+\tau])^p] = \mathbb{E}\left[\left(\int_t^{t+\tau} \exp(\gamma V_s - \frac{\gamma^2}{2}\operatorname{Var} V_s)ds\right)^p\right].$$

- Thus, controlling $\mathscr{E}_T$ allows us to measure how much an approximation of V by $\overline{V}$ impacts the local multifractality of the induced Gaussian measure ξ.

2. A comparison theorem for integrated Volterra processes

$$\mathbb{E}[\phi(X_T)] - \mathbb{E}[\phi(\overline{X}_T)] = ?$$

- Usually, weak error expansions (e.g, for Euler scheme) are done with the **Itô formula** and the **Kolomogorv PDE** satisfied by $u(t,x) = \mathbb{E}[\phi(X_T^{t,x})]$.
- In this case, not possible because V_t is **not a semimartingale** in general and **not a Markov process**
- We will use a recent technique to obtain martingales and recover Markovianity, at the price of the **extension** of the domain of the x variable to a **functionnal space**

- We consider the filtration $\mathscr{F}_s = \sigma(W_r\,;\, r \in [0,s])$ for $s \in \mathbb{T}$. The orthogonal decomposition from [Viens and Zhang, 2019] then writes:

$$\forall s \geq t \in \mathbb{T}, \qquad V_s = \underbrace{\int_0^t K(s-r)dW_r}_{\Theta_s^t \in \mathscr{F}_t} + \underbrace{\int_t^s K(s-r)dW_r}_{I_s^t \perp\!\!\!\perp \mathscr{F}_t}.$$

- We derive that for any test function $\phi : \mathbb{R} \to \mathbb{R}$,

$$\begin{aligned} \mathbb{E}[\phi(X_T)|\mathscr{F}_t] &= \mathbb{E}\left[\phi\left(X_t + \int_t^T b(V_s)ds\right)|\mathscr{F}_t\right] \\ &= \mathbb{E}\left[\phi\left(X_t + \int_t^T b(\Theta_s^t + I_s^t)ds\right)|\mathscr{F}_t\right] \\ &= u(t, X_t, \Theta_{[t,T]}^t), \end{aligned}$$

where $u(t,x,\omega) = \mathbb{E}_{t,x,\omega}[\phi(X_T)] = \mathbb{E}[\phi(X_T)|X_t = x, \Theta_{[t,T]}^t = \omega]$ for all $(t,x,\omega) \in \mathbb{T} \times \mathbb{R} \times C([t,T],\mathbb{R})$.

Let $\Lambda = \mathbb{T} \times \mathbb{R} \times C^0([t,T],\mathbb{R})$, and $u : \Lambda \to \mathbb{R}$. For $\omega \in C^0([t,T])$ we set $\|\omega\|_{\mathbb{T}} = \sup_{s\in[t,T]} |\omega_s|$. We define the **path derivative** of u in ω in the direction $\eta \in C([t,T])$ by:

$$\langle \partial_\omega u(t,x,\omega), \eta \rangle = \lim_{\varepsilon \longrightarrow 0} \frac{u(t,x,\omega+\varepsilon\eta) - u(t,x,\omega)}{\varepsilon},$$

which is a **Gâteaux** derivative, and we define in the same way the second path derivative $\left\langle \partial^2_\omega u(t,x,\omega), (\eta,\zeta) \right\rangle$.

Definition -

We say that u belongs to $C^{2,2}_+(\Lambda)$ if u is C^2 with respect to x and two times **Fréchet differentiable** with respect to ω, and if there exists $k_u, q_u > 0$ such that

$$|\langle \partial_\omega u(t,x,\omega), \eta \rangle| \lesssim (1 + |x|^{k_u} + e^{q_u \|\omega\|_{\mathbb{T}}}) \, \|\eta\|_{\mathbb{T}}$$
$$\left|\left\langle \partial^2_\omega u(t,x,\omega), (\eta,\zeta) \right\rangle\right| \lesssim (1 + |x|^{k_u} + e^{q_u \|\omega\|_{\mathbb{T}}}) \, \|\eta\|_{\mathbb{T}} \, \|\zeta\|_{\mathbb{T}}.$$

Let $u \in C_+^{2,2}(\Lambda)$ and $K^s(r) = K(r-s)$ for $r \geq s$. We have the **functional Itô formula** from [Viens and Zhang, 2019]:

$$u(t,X_t,\Theta^t) - u(0,x,0) = \int_0^t \Bigg(\partial_t u(s,X_s,\Theta^s) + b(V_s)\partial_x u(s,X_s,\Theta^s) + \frac{1}{2}\left\langle \partial^2_\omega u(s,X_s,\Theta^s),(K^s,K^s)\right\rangle \Bigg) ds + \int_0^t \langle \partial_\omega u(s,X_s,\Theta^s),K^s\rangle \, dW_s.$$

And we have the following **path-dependent PDE** for u:

Theorem (Thm 2.25 dans [Bonesini et al., 2023])

Under regularity hypothesis for b and ϕ (e.g, C^2 and polynomial growth for ϕ, C^2 and exponential growth for b):

$$\partial_t u(t,x,\omega) + b(\omega_t)\partial_x u(t,x,\omega) + \frac{1}{2}\left\langle \partial^2_\omega u(t,x,\omega),(K^t,K^t)\right\rangle = 0$$

$$\text{with terminal condition} \quad u(T,x,\omega) = \phi(x),$$

- We introduce the **flow process**:

$$
\begin{aligned}
X_T^{t,x,\omega} &= x + \int_t^T b(V_s^{t,\omega})ds \\
V_s^{t,\omega} &= \omega_s + \int_t^s K(s-r)dW_r, \quad (t,x,\omega) \in \Lambda \text{ and } s \geq t.
\end{aligned}
$$

- By some Gaussian computations, we get the **moment bounds**:

Lemma

Assume that $|b(x)| \leq 1 + e^{k_b x}$ for some $k_b > 0$. Then for all $p \geq 1$,

$$
\sup_{s\in[t,T]} \mathbb{E}[\exp(pV_s^{t,\omega})] \lesssim e^{p\|\omega\|_{\mathbb{T}}} \quad \mathbb{E}[|X_T^{t,x,\omega}|^p] \lesssim |x|^p + e^{2pk_b\|\omega\|_{\mathbb{T}}}
$$

(2) - Regularity of u: representation of the derivatives

We will use the following formal notations:

$$\langle \partial_\omega X_T^{t,x,\omega}, \eta \rangle = \int_t^T b'(s, V_s^{t,\omega}) \eta_s ds.$$

$$\langle \partial_\omega^2 X_T^{t,x,\omega}, (\eta, \zeta) \rangle = \int_t^T b''(s, V_s^{t,\omega}) \eta_s \zeta_s ds.$$

Proposition

Under the same regularity conditions on ϕ and b as before, the function $u : (t,x,\omega) \mapsto \mathbb{E}[\phi(X_T^{t,x,\omega})]$ belongs to $C_+^{2,2}(\Lambda)$ and for any $\eta, \zeta \in C([t,T])$ we have:

$$\begin{aligned}
\partial_x u(t,x,\omega) =& \mathbb{E}[\phi'(X_T^{t,x,\omega})], \\
\partial_x^2 u(t,x,\omega) =& \mathbb{E}[\phi''(X_T^{t,x,\omega})], \\
\langle \partial_\omega u(t,x,\omega), \eta \rangle =& \mathbb{E}[\phi'(X_T^{t,x,\omega}) \langle \partial_\omega X_T^{t,x,\omega}, \eta \rangle], \\
\langle \partial_\omega \partial_x u(t,x,\omega), (\eta, \zeta) \rangle =& \mathbb{E}[\phi''(X_T^{t,x,\omega}) \langle \partial_\omega X_T^{t,x,\omega}, \eta \rangle \zeta_T], \\
\left\langle \partial_\omega^2 u(t,x,\omega), (\eta, \zeta) \right\rangle =& \mathbb{E}[\phi''(X_T^{t,x,\omega}) \langle \partial_\omega X_T^{t,x,\omega}, \eta \rangle \langle \partial_\omega X_T^{t,x,\omega}, \zeta \rangle] \\
&+ \mathbb{E}_{t,x,\omega}[\phi'(X_T^{t,x,\omega}) \left\langle \partial_\omega^2 X_T^{t,x,\omega}, (\eta, \zeta) \right\rangle].
\end{aligned}$$

Let $\overline{\Theta}^t_s = \int_0^t \overline{K}(s-r)dW_r$. We apply the **functionnal Itô formula** to u and $(t, \overline{X}_t, \overline{\Theta}^t)$, we use the **PPDE** satisfied by u, the bilinearity and symmetry of the map $\left\langle \partial^2_\omega u(t,x,\omega), (\cdot,\cdot) \right\rangle$:

$$
\begin{aligned}
\mathbb{E}[\phi(\overline{X}_T)] - \mathbb{E}[\phi(X_T)] =& \mathbb{E}[u(T, \overline{X}_T, \overline{\Theta}^T)] - \mathbb{E}[u(0,x,0)] \\
=& \mathbb{E} \int_0^T \partial_t u(t, \overline{X}_t, \overline{\Theta}^t) dt \\
&+ \mathbb{E} \int_0^T \left\{ b(\overline{V}^t) \partial_x u(t, \overline{X}_t, \overline{\Theta}^t) + \frac{1}{2} \left\langle \partial^2_\omega u(t, \overline{X}_t, \overline{\Theta}^t), (\overline{K}^t, \overline{K}^t) \right\rangle \right\} dt \\
=& \frac{1}{2} \mathbb{E} \int_0^T \left\langle \partial^2_\omega u(t, \overline{X}_t, \overline{\Theta}^t), (\overline{K}^t - K^t, \overline{K}^t + K^t) \right\rangle dt.
\end{aligned}
$$

Then due to the **probabilistic representation** of u we write the error as:

$$
\begin{aligned}
&\frac{1}{2} \int_0^T \mathbb{E}\left[\phi''(X_T^{t, \overline{X}_t, \overline{\Theta}^t}) \left(\int_t^T b'(V_s^{t, \overline{X}^t})(K - \overline{K})(s-t) ds \right) \left(\int_t^T b'(V_s^{t, \overline{X}^t})(K + \overline{K})(s-t) ds \right) \right] dt \\
&+ \frac{1}{2} \int_0^T \mathbb{E}\left[\phi'(X_T^{t, \overline{X}_t, \overline{\Theta}^t}) \left(\int_t^T b''(V_s^{t, \overline{X}^t})(K^2 - \overline{K}^2)(s-t) ds \right) \right] dt.
\end{aligned}
$$

By **pushing the expectations** inside the Lebesgues integrals in the last formula, by the **triangle inequality**, the **growth control** of the coefficients and the **moment bounds** on the stochastic terms, we obtain the following result:

Théorème - Weak comparison theorem

$$|\mathbb{E}[\phi(\overline{X}_T)] - \mathbb{E}[\phi(X_T)]| \lesssim \int_0^T \int_0^t |(K - \overline{K})(s)| ds\, dt + \int_0^T \int_0^t |(K^2 - \overline{K}^2)(s)| ds\, dt. \tag{1}$$

- **Remark:** With additional regularity on the coefficients, a **second use** of the functional Itô formula allows to expand the second term in the RHS of (1) into:

$$\int_0^T \left| \int_0^t K^2(s) ds - \int_0^t \overline{K}^2(s) ds \right| dt + \int_0^T \int_0^t \left| (K^2(s) - \overline{K}^2(s)) \int_0^s (K^2(u) - \overline{K}^2(u)) du \right| ds\, dt.$$

  It is still an **open question** whereas we can get rid of the pink term when K is singular.

3. Applications

The comparison theorem may help to determine which sequences of kernels are the most **suitable approximations** of the fractionnal one in our **modelling problem**. In particular:

- Let $K(r) = r^{H-\frac{1}{2}}$ and $\overline{K}(r) = (r+\tau)^{H-\frac{1}{2}}$. Then $|\mathbb{E}[\phi(\overline{X}_T)] - \mathbb{E}[\phi(X_T)]| \underset{H\to 0}{\lesssim} \frac{T}{2H} + o(\frac{\tau}{2H})$.
- Let $K(r) = r^{H-\frac{1}{2}}$ and $\overline{K}(r) = r^{H-\frac{1}{2}}\mathbf{1}_{\{r\geq\tau\}} + \frac{\tau^{H-\frac{1}{2}}}{\sqrt{2H}}\mathbf{1}_{\{r<\tau\}}$. Then $|\mathbb{E}[\phi(\overline{X}_T)] - \mathbb{E}[\phi(X_T)]| \underset{H\to 0}{\lesssim} \tau\frac{T}{2H} + o(\frac{\tau}{2H})$ **but only if we get rid of the pink term**.

This suggests that there *might* be **better approximations** of the fractionnal than $\overline{K}(r) = (r+\tau)^{H-\frac{1}{2}}$ when H is small, but this is still an open question for now.

- **Goal**: for V a Volterra process with non-trivial kernel, approximate the **path** of V to compute the statistics of the integrated model by Monte-Carlo method:

$$\mathbb{E}\left[\phi\left(\int_0^T b(V_s)ds\right)\right] \simeq \frac{1}{N}\sum_{i=1}^{N}\phi\left(\sum_{j=1}^{n-1}(t_{j+1}-t_j)b(V_{t_j}^{(i)})\right).$$

- This can be achieved by **Markovian approximation** of V when the kernel K is **completely monotone**, i.e if there exists a positive, non decreasing function $\lambda : \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that:

$$K(r) = \int_0^{+\infty} e^{-(t-s)x}\lambda(x)dx.$$

- **Example:** the kernels $K(r) = r^{H-\frac{1}{2}}$ and $K(r) = (r+\tau)^{H-\frac{1}{2}}$ are completely monotones, with associated λ respectively equals to $\lambda(x) = x^{-H-\frac{1}{2}}$ and $\lambda(x) = e^{-\tau x}x^{-H-\frac{1}{2}}$.

Applying the **stochastic Fubini theorem** and discretising the **Laplace transform** of K,

$$\begin{aligned}\int_0^t K(t-s)dW_s &= \int_0^t \left(\int_0^{+\infty} e^{-(t-s)x}\lambda(x)dx\right) dW_s \\ &= \int_0^{+\infty} \lambda(x)dx \left(\int_0^t e^{-(t-s)x} dW_s\right) \\ &\simeq \sum_{i=1}^m w_i Y_t^{x_i},\end{aligned}$$

where

- $(w_i, x_i)_{\{1\le i\le m\}}$ is an appropriate **Gauss quadrature** of order m for $\int_0^{+\infty} f(x)\lambda(x)dx$,
- $(Y_t^{x_i})_{t\in[0,T]}$ is a (**Markov**) **Ornstein-Uhlenbeck** process starting from zero :

$$\begin{aligned} dY_t^{x_i} &= -x_i Y_t^{x_i} dt + dW_t \\ Y_0^{x_i} &= 0.\end{aligned}$$

- Observe that we can write

$$\sum_{i=1}^{m} w_i Y_t^{x_i} = \int_0^t K_m(t-s)dW_s \text{ with } K_m(r) = \sum_{i=1}^{m} w_i e^{-rx_i},$$

- If $(w_i, x_i)_{\{1\leq i\leq m\}}$ comes from a **Gaussian quadrature method**, it follows from Corollary D.2 in [Bayer and Breneis, 2023] that for every m and r, we have

$$K_m(r) \leq K(r),$$

hence $|K^2(r) - K_m^2(r)| = K^2(r) - K_m^2(r)$.

- Applying the **comparison theorem** we derive that

$$|\mathbb{E}[\phi(X_T)] - \mathbb{E}[\phi(X_T^{(m)})]| \leq \int_0^T \int_0^t (K^2(s) - K_m^2(s))ds\, dt$$

where $X_T^{(m)} = \int_0^T b\big(\int_0^s K_m(s-r)dW_r\big)ds$.

- For example, if $K(r) = r^{H-\frac{1}{2}} = \int_0^{+\infty} x^{-\frac{1}{2}-H} e^{-rx}dx$, the weak error from the Markovian approximation is mainly controlled by the **leftover term**

$$\int_{t_0}^T \int_t^T \left(\int_{x_m}^{+\infty} x^{-\frac{1}{2}-H} e^{-(s-t)x}dx\right)^2 dsdt \lesssim \frac{x_m^{-2H}}{2H}.$$

- Using the **functionnal Itô formula** from [Viens and Zhang, 2019] and the **PPDEs** from [Bonesini et al., 2023], we obtain a **comparison theorem** between (Lebesgue) integrated Volterra processes in terms of **deterministic integral differences of their kernels**.

- The error obtained might be improved if we can push the absolute value outside one of the integrals, which would be **crucial for the applications** we have in mind. This is work in progress.

- One may also think to **extend** the result to **stochastic integrals** of Volterra processes, with respect to a possibly correlated other Brownian motion.

Thanks for your attention!

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