On the strong convergence of the ε-EM scheme for time-inhomogeneous jump driven SDEs

08/11/2023

Paul Maurer[1]

Calisto Team

Inria – Université Côte d'Azur

[1] This is a joint work with Mireille Bossy.

- We are interested in the **approximation** and the **simulation** of the solution process $(X_t, t \in [0,T])$ of the SDE

$$X_t = \int_0^t b(X_s)ds + \int_0^t \int_{-\infty}^{\infty} c(s, X_{s^-}, z)\widetilde{N}(ds, dz),$$

where $\widetilde{N}$ is a compensated **random Poisson measure**, with **time-inhomogeneous** compensator measure $\nu_s(dz)ds$, and b and c are deterministic Lipschitz-in-space coefficients.

- We are interested in the **approximation** and the **simulation** of the solution process $(X_t, t \in [0,T])$ of the SDE

$$X_t = \int_0^t b(X_s)ds + \int_0^t \int_{-\infty}^{\infty} c(s, X_{s^-}, z)\widetilde{N}(ds, dz),$$

  where $\widetilde{N}$ is a compensated **random Poisson measure**, with **time-inhomogeneous** compensator measure $\nu_s(dz)ds$, and b and c are deterministic Lipschitz-in-space coefficients.

- This class of SDE is useful to obtain **non-gaussian** stochastic models that may have **several time regimes**. Such a model can be for example used
  - to describe the population dynamics of parasitoid insects (see [BCP+23])
  - to capture option prices over a range of different maturities and strikes (see [CT04])
  - to represent the angle dynamics of non-spherical particles in a turbulent flow (see [CBB22]).

- We are interested in the **approximation** and the **simulation** of the solution process $(X_t, t \in [0,T])$ of the SDE

$$X_t = \int_0^t b(X_s)ds + \int_0^t \int_{-\infty}^{\infty} c(s, X_{s^-}, z)\widetilde{N}(ds, dz),$$

  where $\widetilde{N}$ is a compensated **random Poisson measure**, with **time-inhomogeneous** compensator measure $\nu_s(dz)ds$, and b and c are deterministic Lipschitz-in-space coefficients.

- This class of SDE is useful to obtain **non-gaussian** stochastic models that may have **several time regimes**. Such a model can be for example used
  - to describe the population dynamics of parasitoid insects (see [BCP+23])
  - to capture option prices over a range of different maturities and strikes (see [CT04])
  - to represent the angle dynamics of non-spherical particles in a turbulent flow (see [CBB22]).

- Our goal is to construct an algorithm $(\overline{X}_n)$ to simulate the process (X_t) and obtain **rates of convergence** for the probabilistic **strong error** (trajectorial) between $\overline{X}$ and X.

- Numerical algorithms to approximate SDEs driven by a Brownian motion W are well known in the literature:

$$X_t = \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s.$$

- The simplest method is given by the Euler-Maruyama scheme:

$$\overline{X}_{t_{i+1}} = \overline{X}_{t_i} + b(\overline{X}_{t_i})(t_{i+1} - t_i) + \sigma(\overline{X}_{t_i})(W_{t_{i+1}} - W_{t_i}),$$

  where $(t_i = \frac{iT}{n}, i = 0, \dots, n)$ are the discretization steps, and $n \in \mathbb{N}$.

- This scheme is straightforward to implement because one knows how to simulate the law of any increment $W_s - W_u$ of the Brownian motion easily.

- A first generalisation consists in replacing the driving Brownian motion by a **Lévy** process L, i.e a process L having **independent** and **stationary** increments:

$$X_t = \int_0^t b(X_s)ds + \int_0^t \sigma(X_{s^-})dL_s.$$

- In this case, one may still define the Euler-Maruyama scheme:

$$\overline{X}_{t_{i+1}} = \overline{X}_{t_i} + b(\overline{X}_{t_i})(t_{i+1} - t_i) + \sigma(\overline{X}_{t_i})(L_{t_{i+1}} - L_{t_i}),$$

where $(t_i = \frac{iT}{n}, i = 0, \ldots, n)$ are the discretization steps, and $n \in \mathbb{N}$.

- However, simulating the increments $L_s - L_u$ of the Lévy process is not something easy in general. It is possible for some particular example, such as the so called α-stable process L_α.

- A first generalisation consists in replacing the driving Brownian motion by a **Lévy** process L, i.e a process L having **independent** and **stationary** increments:

$$X_t = \int_0^t b(X_s)ds + \int_0^t \sigma(X_{s^-})dL_s.$$

- In this case, one may still define the Euler-Maruyama scheme:

$$\overline{X}_{t_{i+1}} = \overline{X}_{t_i} + b(\overline{X}_{t_i})(t_{i+1} - t_i) + \sigma(\overline{X}_{t_i})(L_{t_{i+1}} - L_{t_i}),$$

where $(t_i = \frac{iT}{n}, i = 0, \ldots, n)$ are the discretization steps, and $n \in \mathbb{N}$.

- However, simulating the increments $L_s - L_u$ of the Lévy process is not something easy in general. It is possible for some particular example, such as the so called α-stable process L_α.

- Thanks to the **Lévy-Itô décomposition**, if L has at least a moment of order 2, it can be written

$$L_t = \int_0^t \int_{-\infty}^{\infty} z(N(ds,dz) - \nu(dz)ds) = \int_0^t \int_{-\infty}^{\infty} z\widetilde{N}(ds,dz),$$

where ν is a deterministic measure called the **Lévy measure** of L.

- A second generalisation consists in replacing the driving Brownian motion by an **additive** process A, i.e a process A having **independent** increments:

$$X_t = \int_0^t b(X_s)ds + \int_0^t \sigma(X_{s^-})dA_s.$$

- In this case, one may still define the Euler-Maruyama scheme:

$$\overline{X}_{t_{i+1}} = \overline{X}_{t_i} + b(\overline{X}_{t_i})(t_{i+1} - t_i) + \sigma(\overline{X}_{t_i})(A_{t_{i+1}} - A_{t_i}),$$

where $(t_i = \frac{iT}{n}, i = 0, \ldots, n)$ are the discretization steps, and $n \in \mathbb{N}$.

- However, simulating the increments $A_s - A_u$ of the additive process is not something easy in general.

- Thanks to the **Lévy-Itô décomposition**, if A has at least a moment of order 2, it can be written

$$A_t = \int_0^t \int_{-\infty}^{\infty} z(N(ds,dz) - \nu_s(dz)ds) = \int_0^t \int_{-\infty}^{\infty} z\widetilde{N}(ds,dz),$$

where $(\nu_s)_{s \in [0,T]}$ is a collection of Lévy measures.

- Finally, based on the Lévy-Itô decomposition, one may think of an **"increments-free"** generalisation, leading to the SDE:

$$X_t = \int_0^t b(X_s)ds + \int_0^t \int_{-\infty}^{\infty} c(X_{s^-}, z)\widetilde{N}(ds, dz).$$

- However, in this case we are **not** able to give a sense to the Euler-Maruyama scheme:

$$\overline{X}_{t_{i+1}} = \overline{X}_{t_i} + b(\overline{X}_{t_i})(t_{i+1} - t_i) + ???$$

where $(t_i = \frac{iT}{n}, i = 0, \ldots, n)$ are the discretization steps, and $n \in \mathbb{N}$.

- Indeed, we can't define a discretisation of X that relies on the increments of an underlying stochastic process anymore, except in the case where c can be written $c(x,z) = \sigma(x)f(z)$.

- Let $F:[0,T]\times\mathbb{R}\to\mathbb{R}$ be a measurable function such that $\int_0^T\int_{-\infty}^{\infty}|F(s,z)|^2\nu_s(dz)ds<\infty$. We want to simulate the stochastic integral:

$$I(F)=\int_0^T\int_{-\infty}^{+\infty}F(s,z)\widetilde{N}(ds,dz)$$

2. Approximation of a random Poisson integral

- Let $F : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a measurable function such that $\int_0^T \int_{-\infty}^{\infty} |F(s,z)|^2 \nu_s(dz)ds < \infty$. We want to simulate the stochastic integral:

$$I(F) = \int_0^T \int_{-\infty}^{+\infty} F(s,z)\widetilde{N}(ds,dz)$$

- Taking the threshold $\varepsilon > 0$, $I(F)$ can be separated into it's **large jumps** part $I_l^{\varepsilon}(F)$ and **small jumps** part $I_l^{\varepsilon}(F)$:

$$I_l^{\varepsilon}(F) = \int_0^T \int_{\mathbb{R}\setminus B(\varepsilon)} F(s,z)\widetilde{N}(ds,dz), \qquad I_l^{\varepsilon}(F) = \int_0^T \int_{B(\varepsilon)} F(s,z)\widetilde{N}(ds,dz),$$

$$I(F) = I_l^{\varepsilon}(F) + I_l^{\varepsilon}(F).$$

The large jumps : a direct simulation method

- The large jump integral can be represented by the difference of a **finite random sum** and a deterministic integral:

$$\int_0^T \int_{\mathbb{R}\setminus B(\varepsilon)} F(s,z)N(ds,dz) = \sum_{j=1}^{N^\varepsilon(T)} F(T^\varepsilon(j), Z^\varepsilon(j)) - \int_0^t \int_{\mathbb{R}\setminus B(\varepsilon)} F(s,z)\nu_s(dz)ds. \qquad (1)$$

The large jumps : a direct simulation method

- The large jump integral can be represented by the difference of a **finite random sum** and a deterministic integral:

$$\int_0^T \int_{\mathbb{R}\backslash B(\varepsilon)} F(s,z) N(ds,dz) = \sum_{j=1}^{N^\varepsilon(T)} F(T^\varepsilon(j), Z^\varepsilon(j)) - \int_0^t \int_{\mathbb{R}\backslash B(\varepsilon)} F(s,z) \nu_s(dz) ds. \quad (1)$$

- In the above formula, N^ε is a (time-inhomogeneous) **Poisson process** with intensity function $\boxed{\lambda^\varepsilon(t) = \int_{\mathbb{R}\backslash B(\varepsilon)} \nu_t(dz)}$, and jump times $\boxed{T^\varepsilon(j) = \inf\{t \in [0,T], N^\varepsilon(t) = j\}}$.

The large jumps : a direct simulation method

- The large jump integral can be represented by the difference of a **finite random sum** and a deterministic integral:

$$\int_0^T \int_{\mathbb{R}\backslash B(\varepsilon)} F(s,z)N(ds,dz) = \sum_{j=1}^{N^\varepsilon(T)} F(T^\varepsilon(j), Z^\varepsilon(j)) - \int_0^t \int_{\mathbb{R}\backslash B(\varepsilon)} F(s,z)\nu_s(dz)ds. \quad (1)$$

- In the above formula, N^ε is a (time-inhomogeneous) **Poisson process** with intensity function $\boxed{\lambda^\varepsilon(t) = \int_{\mathbb{R}\backslash B(\varepsilon)} \nu_t(dz)}$, and jump times $\boxed{T^\varepsilon(j) = \inf\{t \in [0,T], N^\varepsilon(t) = j\}}$.
- The random variables $Z^\varepsilon(j)$ for $j \leq N^\varepsilon(T)$ have conditional distribution given the jump times given by:

$$\boxed{\forall B \in \mathscr{B}(\mathbb{R}) \qquad \mathbb{P}(Z^\varepsilon(j) \in B \mid T^\varepsilon(j) = t) = \frac{\nu_t(B \cap \mathbb{R}\backslash B(\varepsilon))}{\nu_t(\mathbb{R}\backslash B(\varepsilon))}}.$$

The large jumps : a direct simulation method

- The large jump integral can be represented by the difference of a **finite random sum** and a deterministic integral:

$$\int_0^T \int_{\mathbb{R}\setminus B(\varepsilon)} F(s,z)N(ds,dz) = \sum_{j=1}^{N^\varepsilon(T)} F(T^\varepsilon(j), Z^\varepsilon(j)) - \int_0^t \int_{\mathbb{R}\setminus B(\varepsilon)} F(s,z)\nu_s(dz)ds. \quad (1)$$

- In the above formula, N^ε is a (time-inhomogeneous) **Poisson process** with intensity function $\boxed{\lambda^\varepsilon(t) = \int_{\mathbb{R}\setminus B(\varepsilon)} \nu_t(dz)}$, and jump times $\boxed{T^\varepsilon(j) = \inf\{t \in [0,T], N^\varepsilon(t) = j\}}$.
- The random variables $Z^\varepsilon(j)$ for $j \leq N^\varepsilon(T)$ have conditional distribution given the jump times given by:

$$\boxed{\forall B \in \mathscr{B}(\mathbb{R}) \qquad \mathbb{P}(Z^\varepsilon(j) \in B \mid T^\varepsilon(j) = t) = \frac{\nu_t(B \cap \mathbb{R}\setminus B(\varepsilon))}{\nu_t(\mathbb{R}\setminus B(\varepsilon))}}.$$

- Note that if the time-dependence of ν_t is **multiplicative**, i.e if one has $\nu_t(dz) = \phi(t)\nu(dz)$, then the latter distribution is homogeneous in time. In this case the jump sizes $Z^\varepsilon(j)$ are **i.i.d**.

The large jumps : a direct simulation method

- The large jump integral can be represented by the difference of a **finite random sum** and a deterministic integral:

$$\int_0^T \int_{\mathbb{R}\setminus B(\varepsilon)} F(s,z)N(ds,dz) = \sum_{j=1}^{N^\varepsilon(T)} F(T^\varepsilon(j), Z^\varepsilon(j)) - \int_0^t \int_{\mathbb{R}\setminus B(\varepsilon)} F(s,z)\nu_s(dz)ds. \quad (1)$$

- Hence, to perform a simulation of $I_l^\varepsilon(F)$, one needs:

1. To **generate the Poisson process** N^ε: this can be done with thinning method if the function λ is bounded, or by (eventually numerical) inversion of $t \mapsto \lambda(t)$;
2. To **generate the jump sizes** random variables $Z^\varepsilon(j)$ for any $j \leq N^\varepsilon(T)$: for usual distributions, this can be done by inversion or acceptance-rejection methods;
3. To **compute the deterministic integral** $\int_0^t \int_{\mathbb{R}\setminus B(\varepsilon)} F(s,z)\nu_s(dz)ds$: it can be done analytically or numerically, depending on the difficulty.

The large jumps : a simple (and useful !) example

- We take the example of a 1-truncated α-stable process, i.e $F(s,z) = z$ and

$$\nu_t(dz) = f(t)|z|^{-(1+\alpha)}\mathbb{1}_{\{|z|\leq 1\}},$$

where $\alpha \in (0,2]$ and $f \in L^\infty([0,T])$.

1. The **Poisson process** N^ε has intensity function $\lambda^\varepsilon(t) \leq 2\|f\|_\infty \frac{\varepsilon^{-\alpha}-1}{\alpha}$, allowing to use a thinning method.
2. The **jump sizes** $Z^\varepsilon(j)$ are i.i.d and $Z^\varepsilon(1)$ has explicit quantile function given by

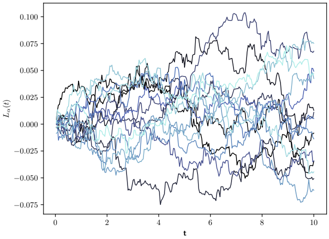
$$\forall y \in]0,1], \quad Q_\varepsilon(y) = \begin{cases} -\{2y(\varepsilon^{-\alpha}-1)+1\}^{-\frac{1}{\alpha}}, & \text{if } y \in (0,\frac{1}{2}], \\ \{(1-2y)(\varepsilon^{-\alpha}-1)+\varepsilon^{-\alpha}\}^{-\frac{1}{\alpha}}, & \text{if } y \in (\frac{1}{2},1]. \end{cases}$$

3. The **deterministic integral** $\int_0^t \int_{\mathbb{R}\setminus B(\varepsilon)} F(s,z)\nu_s(dz)ds$ is straightforward to compute:

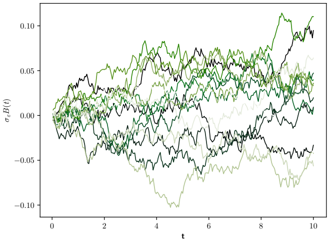
$$\int_0^t \int_{\mathbb{R}\setminus B(\varepsilon)} z\nu_s(dz)ds = \left(\int_0^t f(s)ds\right)\left(\int_{\varepsilon\leq|z|\leq 1} z|z|^{-1-\alpha}dz\right) = 0.$$

The small jumps : an extension of the Asmussen-Rosinski method

- Generally, exact simulation of the small jumps integral $I_l^{\varepsilon}(F)$ is not possible, but we may **approximate** it using the following idea:



Trajectories[2] of a 0.01-truncated $\frac{1}{2}$-stable process L_t, i.e. $\nu(dz) = \mathbb{1}_{\{|z| \leq 0.01\}} \frac{dz}{|z|^{3/2}}$)



Trajectories of a renormalized Brownian motion $\sqrt{L_1} B_t$

[1] Obtained by direct simulation, which is possible in this very specific case thanks to the acceptance-rejection algorithm developped by Dassios, Lim and Qu in [DLQ19]

The small jumps : an extension of the Asmussen-Rosinski method

- Generally, exact simulation of the small jumps integral $I_l^\varepsilon(F)$ is not possible, but we may **approximate** it using the following idea.
- We substitute the stochastic integral $I_l^\varepsilon(F)$ with a **Gaussian** random variable having an **equivalent variance**:

$$\mathscr{L}aw(I_l^\varepsilon(F)) \simeq \left(\int_0^T \int_{B(\varepsilon)} |F(s,z)|^2 \nu_s(dz)ds\right)^{\frac{1}{2}} \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ designates the standard normal distribution.

The small jumps : an extension of the Asmussen-Rosinski method

- Generally, exact simulation of the small jumps integral $I_l^\varepsilon(F)$ is not possible, but we may **approximate** it using the following idea.
- We substitute the stochastic integral $I_l^\varepsilon(F)$ with a **Gaussian** random variable having an **equivalent variance**:

$$\mathscr{L}aw(I_l^\varepsilon(F)) \simeq \left(\int_0^T \int_{B(\varepsilon)} |F(s,z)|^2 \nu_s(dz)ds \right)^{\frac{1}{2}} \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ designates the standard normal distribution.

- To **quantify** the error made in this approximation, we will use the **Wasserstein distance** of order p defined by

$$\mathscr{W}_p(\mathscr{L}_1,\mathscr{L}_2) = \inf_{(X_1,X_2)\in\pi(\mathscr{L}_1,\mathscr{L}_2)} \mathbb{E}[|X_1 - X_2|^p]^{\frac{1}{p}},$$

where $(X_1,X_2) \in \pi(\mathscr{L}_1,\mathscr{L}_2)$, means that the random variables X_1, X_2 verify $\mathscr{L}aw(X_i) = \mathscr{L}_i$.

The small jumps : an extension of the Asmussen-Rosinski method

Proposition 1

Let $p \geq 1$. Assume that $F(t,\cdot)$ is non identically zero on $B(1)$ for any $t \in [0,T]$ and

$$\int_0^T \int_{B(1)} |F(s,z)|^{p+2} \nu_s(dz)ds \; + \int_0^T \left(\int_{B(1)} |F(s,z)|^2 \nu_s(dz) \right)^{\frac{p}{2}+1} ds \; < \infty.$$

Then there exists a constant $\mathscr{A}(p)$, only depending on p, such that for every $\varepsilon \in (0,1)$, the following inequality holds for any $t \in [0,T]$:

$$\begin{aligned} &\mathscr{W}_p \left(\mathscr{L}aw\left(\int_0^t \int_{B(\varepsilon)} F(s,z)\widetilde{N}(ds,dz) \right), \mathscr{N}\left(0, \int_0^t \int_{B(\varepsilon)} |F(s,z)|^2 \nu_s(dz)ds \right) \right) \\ &\leq \mathscr{A}(p) \left(\frac{\int_0^t \int_{B(\varepsilon)} |F(s,z)|^{p+2} \nu_s(dz)ds}{\int_0^t \int_{B(\varepsilon)} |F(s,z)|^2 \nu_s(dz)ds} \right)^{\frac{1}{p}}. \end{aligned} \qquad (1)$$

- The term in the right hand-side goes to zero when ε goes to zero on good conditions on F (a sufficient condition is that $\lim_{|z|\to 0} \sup_{s\in[0,T]} |F(s,z)| = 0$).

Idea of the proof

- The proof relies on a W_p-distance quantification of the convergence of the CLT (Rio's conjecture, proved by Bobkov in 2018 in [Bob18]):

Theorem – (S.G Bobkov, 2018)

For $p \geq 1$, there exists $c_p > 0$ depending only on p such that if $X_1, \ldots, X_m$ are independent random variables with $\sum_{j=1}^m \mathrm{Var}(X_j) = 1$, then

$$\mathscr{W}_p\left(\mathscr{L}aw\left(\sum_{j=1}^m X_j\right), \mathcal{N}(0,1)\right) \leq c_p \left(\sum_{j=1}^m \mathbb{E}[|X_j|^{p+2}]\right)^{\frac{1}{p}}.$$

- We apply this result to the independent random variables

$$X_j = \int_{\tau_{j-1}}^{\tau_j} \int_{B(\varepsilon)} F(s,z)\widetilde{N}(ds,dz), \quad j \in \{1,\ldots,m\},$$

where $\tau_j = \frac{jt}{m}$, and estimate the $p+2$-moment of X_j using **Kunita inequality** for random Poisson integrals, that we will recall later in this talk.

- This discussion allows us to define what we call the **ε-Euler Maruyama** scheme to approximate the process X_t in introduction.
- We fix a threshold $\varepsilon \in (0,1)$. For $n \in \mathbb{N}^*$, we define $0 = t_0 < \cdots < t_n = T$, a discretisation of the interval $[0,T]$ with constant steps, i.e $t_i = i\frac{T}{n}$. Let $(\xi_i)_{i\in\{1,\ldots,n\}}$ a sequence of i.i.d standard Gaussian random variables. We define $\overline{X}^\varepsilon$ by $\overline{X}^\varepsilon_0 = 0$ and

$$\begin{aligned}\overline{X}^\varepsilon_{t_i} =& \overline{X}^\varepsilon_{t_{i-1}} + b(t_{i-1}, \overline{X}^\varepsilon_{t_{i-1}})\frac{T}{n} - \int_{t_{i-1}}^{t_i}\int_{\mathbb{R}\setminus B(\varepsilon)} c(s, \overline{X}^\varepsilon_{t_{i-1}}, z)\nu_s(dz)ds \\ &+ \left(\int_{t_{i-1}}^{t_i}\int_{B(\varepsilon)} c^2(s, \overline{X}^\varepsilon_{t_{i-1}}, z)\nu_s(dz)ds\right)^{\frac{1}{2}}\xi_i + \sum_{j=N^\varepsilon(t_{i-1})+1}^{N^\varepsilon(t_i)} c(T^\varepsilon(j), \overline{X}^\varepsilon_{t_{i-1}}, Z^\varepsilon(j)).\end{aligned}$$

- This discussion allows us to define what we call the **ε-Euler Maruyama** scheme to approximate the process X_t in introduction.

- We fix a threshold $\varepsilon \in (0,1)$. For $n \in \mathbb{N}^*$, we define $0 = t_0 < \cdots < t_n = T$, a discretisation of the interval $[0,T]$ with constant steps, i.e $t_i = i\frac{T}{n}$. Let $(\xi_i)_{i\in\{1,\dots,n\}}$ a sequence of i.i.d standard Gaussian random variables. We define $\overline{X}^\varepsilon$ by $\overline{X}^\varepsilon_0 = 0$ and

$$
\begin{aligned}
\overline{X}^\varepsilon_{t_i} =& \overline{X}^\varepsilon_{t_{i-1}} + b(t_{i-1}, \overline{X}^\varepsilon_{t_{i-1}})\frac{T}{n} - \int_{t_{i-1}}^{t_i}\int_{\mathbb{R}\setminus B(\varepsilon)} c(s, \overline{X}^\varepsilon_{t_{i-1}}, z)\nu_s(dz)ds \\
&+ \left(\int_{t_{i-1}}^{t_i}\int_{B(\varepsilon)} c^2(s, \overline{X}^\varepsilon_{t_{i-1}}, z)\nu_s(dz)ds\right)^{\frac{1}{2}}\xi_i + \sum_{j=N^\varepsilon(t_{i-1})+1}^{N^\varepsilon(t_i)} c(T^\varepsilon(j), \overline{X}^\varepsilon_{t_{i-1}}, Z^\varepsilon(j)).
\end{aligned}
$$

- We will know be giving a convergence result for $\overline{X}^\varepsilon$ in the L^p-norm. Note that this convergence will depend on **two parameters**, which are the number n of discretisation steps and the small jumps/big jumps threshold ε.

4. Hypothesis required for L^p-strong convergence

We fix $p \geq 2$ and $T > 0$. We consider a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in\mathbb{R}_+})$ equipped with a standard Brownian motion B and a random Poisson measure N.

- **(H1) - Regularity.** We assume that there exists constants L_a, L_b and a measurable function $L_c : [0,T] \times \mathbb{R} \to \mathbb{R}_+$

$$
\begin{aligned}
|b(x) - b(y)| &\leq L_b \ (|x-y|), & x, y \in \mathbb{R}, \\
|c(t,x,z) - c(t,y,z)| &\leq L_c(t,z)\ |x-y|, & x, y \in \mathbb{R}, t \in [0,T], z \in \mathbb{R}.
\end{aligned}
$$

4. Hypothesis required for L^p-strong convergence

We fix $p \geq 2$ and $T > 0$. We consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in \mathbb{R}_+})$ equipped with a standard Brownian motion B and a random Poisson measure N.

- **(H1) - Regularity.** We assume that there exists constants L_a, L_b and a measurable function $L_c : [0,T] \times \mathbb{R} \to \mathbb{R}_+$

$$|b(x) - b(y)| \leq L_b \ (|x-y|), \qquad x, y \in \mathbb{R},$$
$$|c(t,x,z) - c(t,y,z)| \leq L_c(t,z) \ |x-y|, \qquad x, y \in \mathbb{R}, t \in [0,T], z \in \mathbb{R}.$$

- **(H2) - Integrability.** We assume that the function ψ_p defined by

$$\psi_p(t) = \left(\int_{-\infty}^{\infty} |L_c(t,z) \vee |c(t,0,z)||^2 \, \nu_t(dz) \right)^{p/2} + \int_{-\infty}^{\infty} |L_c(t,z) \vee |c(t,0,z)||^p \, \nu_t(dz)$$

belongs to $L^{1+\zeta}([0,T])$ for some $\zeta \in (0,1]$.

4. Hypothesis required for L^p-strong convergence

We fix $p \geq 2$ and $T > 0$. We consider a filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, (\mathscr{F}_t)_{t\in\mathbb{R}_+})$ equipped with a standard Brownian motion B and a random Poisson measure N.

- **(H1) - Regularity.** We assume that there exists constants L_a, L_b and a measurable function $L_c : [0,T] \times \mathbb{R} \to \mathbb{R}_+$

$$\begin{aligned} |b(x) - b(y)| &\leq L_b \; (|x-y|), && x, y \in \mathbb{R}, \\ |c(t,x,z) - c(t,y,z)| &\leq L_c(t,z) \; |x-y|, && x, y \in \mathbb{R}, t \in [0,T], z \in \mathbb{R}. \end{aligned}$$

- **(H2) - Integrability.** We assume that the function ψ_p defined by

$$\psi_p(t) = \left(\int_{-\infty}^{\infty} |L_c(t,z) \vee |c(t,0,z)||^2 \, \nu_t(dz) \right)^{p/2} + \int_{-\infty}^{\infty} |L_c(t,z) \vee |c(t,0,z)||^p \, \nu_t(dz)$$

  belongs to $L^{1+\zeta}([0,T])$ for some $\zeta \in (0,1]$.

- **(H3) - A.R. Approximation.** We assume that there exists $\varepsilon^* \in (0,1]$ such that
  - **(Moments)** for all $x \in \mathbb{R}$ and $t \in [0,T]$, $c(t,x,\cdot)$ is not identically zero on $B(\varepsilon^*)$ and

$$\int_0^T \int_{B(\varepsilon^*)} |c(t,x,z)|^{p+2} \nu_t(dz) dt + \int_0^T \left(\int_{B(\varepsilon^*)} |c(t,x,z)|^2 \nu_t(dz) \right)^{\frac{p}{2}+1} dt < \infty.$$

  - **(Coupling)** for all $x \in \mathbb{R}$ and $t \in [0,T]$, the image measure of $\mathbb{1}_{\{z \in B(\varepsilon^*)\}} \nu_t(dz)$ by $z \mapsto c(t,x,z)$ has a density with respect to the Lebesgue measure on $\mathbb{R}$ and satisfies $\int_{B(\varepsilon^*)} \nu_t(dz) = \infty$.

Théorème 1 – (Bossy, Maurer 2023)

We assume that **(H1)**, **(H2)** and **(H3)** hold.

$(i.)$ For any $\varepsilon \in (0, \varepsilon^*]$, there exists a sequence $(\widehat{X}^{\varepsilon}_{t_i})_{i \in \{0,\dots,n\}}$ of random variables on $(\Omega, \mathscr{F})$, such that for any $i \in \{0, \dots, n\}$, $\overline{X}^{\varepsilon}_{t_i}$ is $\mathscr{F}_{t_i}$- measurable, and verifies Law$(\widehat{X}^{\varepsilon}_{t_i})$ = Law$(\overline{X}^{\varepsilon}_{t_i})$. Moreover, there exists $\overline{m}(p,T) > 0$ such that

$$\sup_{\varepsilon \in [0,\varepsilon^*)} \mathbb{E}\Big[\sup_{i \in \{0,\dots,n\}} |\widehat{X}^{\varepsilon}_{t_i}|^p\Big] \leq \overline{m}(p,T).$$

Théorème 1 – (Bossy, Maurer 2023)

We assume that **(H1)**, **(H2)** and **(H3)** hold.

$(i.)$ For any $\varepsilon \in (0, \varepsilon^*]$, there exists a sequence $(\widehat{X}^{\varepsilon}_{t_i})_{i \in \{0,\dots,n\}}$ of random variables on $(\Omega, \mathcal{F})$, such that for any $i \in \{0, \dots, n\}$, $\overline{X}^{\varepsilon}_{t_i}$ is $\mathcal{F}_{t_i}$- measurable, and verifies Law$(\widehat{X}^{\varepsilon}_{t_i}) =$ Law$(\overline{X}^{\varepsilon}_{t_i})$. Moreover, there exists $\overline{m}(p,T) > 0$ such that

$$\sup_{\varepsilon \in [0,\varepsilon^*)} \mathbb{E}\Big[\sup_{i \in \{0,\dots,n\}} |\widehat{X}^{\varepsilon}_{t_i}|^p \Big] \leq \overline{m}(p,T).$$

$(ii.)$ The following inequality stands true for any $\varepsilon \in (0, \varepsilon^*]$:

$$\Big\| \sup_{i \in \{0,\dots,n\}} \big| X_{t_i} - \widehat{X}^{\varepsilon}_{t_i} \big| \Big\|_{L^p(\Omega)} \preccurlyeq n^{-\left\{ \frac{2\zeta}{p(1+\zeta)} \right\}} + \delta^n_p(\varepsilon),$$

Théorème 1 – (Bossy, Maurer 2023)

We assume that **(H1)**, **(H2)** and **(H3)** hold.

$(i.)$ For any $\varepsilon \in (0,\varepsilon^*]$, there exists a sequence $(\widehat{X}^{\varepsilon}_{t_i})_{i\in\{0,\ldots,n\}}$ of random variables on $(\Omega,\mathcal{F})$, such that for any $i \in \{0,\ldots,n\}$, $\overline{X}^{\varepsilon}_{t_i}$ is $\mathcal{F}_{t_i}$- measurable, and verifies Law$(\widehat{X}^{\varepsilon}_{t_i})$ = Law$(\overline{X}^{\varepsilon}_{t_i})$. Moreover, there exists $\overline{m}(p,T) > 0$ such that

$$\sup_{\varepsilon\in[0,\varepsilon^*)} \mathbb{E}\Big[\sup_{i\in\{0,\ldots,n\}} |\widehat{X}^{\varepsilon}_{t_i}|^p\Big] \leq \overline{m}(p,T).$$

$(ii.)$ The following inequality stands true for any $\varepsilon \in (0,\varepsilon^*]$:

$$\Big\|\sup_{i\in\{0,\ldots,n\}} |X_{t_i} - \widehat{X}^{\varepsilon}_{t_i}|\Big\|_{L^p(\Omega)} \preccurlyeq n^{-\left\{\frac{2\zeta}{p(1+\zeta)}\right\}} + \delta^n_p(\varepsilon),$$

where
$$\delta^n_p(\varepsilon) = \left(\sum_{k=1}^{n} \mathbb{E}\left[\left(\frac{\int_{t_{k-1}}^{t_k}\int_{B(\varepsilon)} |c(s,\widehat{X}^{\varepsilon}_{t_{k-1}},z)|^{p+2}\nu_s(dz)ds}{\int_{t_{k-1}}^{t_k}\int_{B(\varepsilon)} |c(s,\widehat{X}^{\varepsilon}_{t_{k-1}},z)|^{2}\nu_s(dz)ds}\right)^{\frac{1}{p}}\right]^2\right)^{\frac{1}{2}}$$

satisfies $\lim_{\varepsilon\to 0}\delta^n_p(\varepsilon) = 0$ when the following sufficient condition holds:

$$\lim_{|z|\to 0}\sup_{t\in[0,T]} |L_c(t,z)| = 0.$$

6. Strong convergence of $\overline{X}^{\varepsilon}$ in $L^p(\Omega)$: corollary

Corollaire

We assume in addition to **(H1)**, **(H2)** and **(H3)** that there exists a constant C_T satisfying

$$\forall (s,x,z) \in [0,T] \times \mathbb{R} \times B(\varepsilon^*), \quad |c(t,x,z)| \leq C_T |z| (1+|x|). \tag{1}$$

- Then, for any $\varepsilon \in (0,\varepsilon^*]$, the L^p-strong error of the $((\Omega,\mathscr{F})$-representation of the) ε-EM scheme $\widehat{X}^{\varepsilon}$ satisfies:

$$\Big\| \sup_{i\in\{0,\ldots,n\}} \big| X_{t_i} - \widehat{X}^{\varepsilon}_{t_i} \big| \Big\|_{L^p(\Omega)} \preccurlyeq n^{-\left\{\frac{2\zeta}{p(1+\zeta)}\right\}} + \varepsilon\sqrt{n}.$$

Corollaire

We assume in addition to **(H1)**, **(H2)** and **(H3)** that there exists a constant C_T satisfying

$$\forall (s,x,z) \in [0,T] \times \mathbb{R} \times B(\varepsilon^*), \quad |c(t,x,z)| \leq C_T |z|(1+|x|). \tag{1}$$

- Then, for any $\varepsilon \in (0, \varepsilon^*]$, the L^p-strong error of the $((\Omega, \mathscr{F})$-representation of the) ε-EM scheme $\widehat{X}^{\varepsilon}$ satisfies:

$$\left\| \sup_{i \in \{0,\ldots,n\}} \left| X_{t_i} - \widehat{X}^{\varepsilon}_{t_i} \right| \right\|_{L^p(\Omega)} \preccurlyeq n^{-\left\{ \frac{2\zeta}{p(1+\zeta)} \right\}} + \varepsilon \sqrt{n}.$$

- Moreover, suppose we have $\psi_p \in L^2([0,T])$ (i.e. $\zeta = 1$). With ε taken such that

$$\varepsilon \leq n^{-\left(\frac{1}{2}+\frac{1}{p}\right)} \wedge \varepsilon^*,$$

we obtain the following convergence rate for the L^p-strong error:

$$\left\| \sup_{i \in \{0,\ldots,n\}} \left| X_{t_i} - \widehat{X}^{\varepsilon}_{t_i} \right| \right\|_{L^p(\Omega)} \preccurlyeq n^{-\frac{1}{p}}.$$

The continuous Euler-Peano scheme as a pivot term

- We use as a **pivot term** the SDE with **frozen coefficients** (or Euler-Peano scheme) $\widetilde{X}$ defined by

$$\widetilde{X}_t = \int_0^t b(\widetilde{X}_{\eta(s)})ds + \int_0^t \int_{-\infty}^{+\infty} c(s, \widetilde{X}_{\eta(s^-)}, z)\widetilde{N}(ds, dz),$$

where $\eta(t) = t_i$ if $t \in [t_i, t_{i+1})$.

The continuous Euler-Peano scheme as a pivot term

- We use as a **pivot term** the SDE with **frozen coefficients** (or Euler-Peano scheme) $\widetilde{X}$ defined by

$$\widetilde{X}_t = \int_0^t b(\widetilde{X}_{\eta(s)})ds + \int_0^t \int_{-\infty}^{+\infty} c(s, \widetilde{X}_{\eta(s^-)}, z)\widetilde{N}(ds, dz),$$

where $\eta(t) = t_i$ if $t \in [t_i, t_{i+1})$.

- We first prove a rate of convergence for $\widetilde{X}$ and then compare $\widetilde{X}$ with our scheme:

Proposition 2 - L^p-convergence of the Euler-Peano scheme

Assume **(H1)** and **(H2)**. Then for all $n \in \mathbb{N}^*$,

$$\left\| \sup_{t\in[0,T]} |X_t - \widetilde{X}_t| \right\|_{L^p(\Omega)} \preccurlyeq n^{-\left\{\frac{2\zeta}{p(1+\zeta)}\right\}}$$

The continuous Euler-Peano scheme as a pivot term

- We use as a **pivot term** the SDE with **frozen coefficients** (or Euler-Peano scheme) $\widetilde{X}$ defined by

$$\widetilde{X}_t = \int_0^t b(\widetilde{X}_{\eta(s)})ds + \int_0^t \int_{-\infty}^{+\infty} c(s, \widetilde{X}_{\eta(s^-)}, z)\widetilde{N}(ds, dz),$$

where $\eta(t) = t_i$ if $t \in [t_i, t_{i+1})$.

- We first prove a rate of convergence for $\widetilde{X}$ and then compare $\widetilde{X}$ with our scheme:

Proposition 2 - L^p-convergence of the Euler-Peano scheme

Assume **(H1)** and **(H2)**. Then for all $n \in \mathbb{N}^*$,

$$\left\| \sup_{t\in[0,T]} |X_t - \widetilde{X}_t| \right\|_{L^p(\Omega)} \preccurlyeq n^{-\left\{\frac{2\zeta}{p(1+\zeta)}\right\}}$$

- The proof of Proposition 2 relies on a **Gronwall** argument as it is usually the case for a standard strong convergence proof, but has some specificity due to the **non-continuous paths** of the process and the **time-inhomogeneity** of the jumps.

Kunita inequality

- To prove Proposition 2, we need a tool to estimate the **L^p-moments** of a stochastic Poisson integral. Since these types of integrals are **martingales**, one may want to use the **Burkhölder-Davis-Gundy (BDG)** inequality :

$$\mathbb{E}\left[\sup_{0\le s\le t} |M_s|^p\right] \le C_p^{BDG}\, \mathbb{E}[[M,M]_t^{p/2}],$$

Kunita inequality

- To prove Proposition 2, we need a tool to estimate the **L^p-moments** of a stochastic Poisson integral. Since these types of integrals are **martingales**, one may want to use the **Burkhölder-Davis-Gundy (BDG)** inequality :

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |M_s|^p\right] \leq C_p^{BDG}\, \mathbb{E}[[M,M]_t^{p/2}],$$

- However, in the case of **discontinuous** process, the quadratic variation $[M,M]$ is not equal to the predictable quadratic variation $\langle M,M\rangle$, and we only have tools to estimate the latter.

Kunita inequality

- To prove Proposition 2, we need a tool to estimate the **L^p-moments** of a stochastic Poisson integral. Since these types of integrals are **martingales**, one may want to use the **Burkhölder-Davis-Gundy (BDG)** inequality :

$$\mathbb{E}\left[\sup_{0\le s\le t}|M_s|^p\right]\le C_p^{BDG}\,\mathbb{E}[[M,M]_t^{p/2}],$$

- However, in the case of **discontinuous** process, the quadratic variation $[M,M]$ is not equal to the predictable quadratic variation $\langle M,M\rangle$, and we only have tools to estimate the latter.
- For this reason, we use **another inequality** that is more specific to Poisson integrals:

Lemma (Kunita inequality)

Let F be a predictable stochastic process and $I_t=\int_0^t\int_{-\infty}^{\infty}F(s,z)\widetilde{N}(ds,dz)$. Then for all $p\ge 2$ there exists a constant C depending only on p and T such that

$$\mathbb{E}\left[\sup_{s\in[0,t]}|I_s|^p\right]\le C\int_0^t\mathbb{E}\left[\left(\int_{-\infty}^{\infty}|F(s,z)|^2\nu_s(dz)\right)^{p/2}\right]ds+C\int_0^t\mathbb{E}\left[\int_{-\infty}^{\infty}|F(s,z)|^p\nu_s(dz)\right]ds$$

Proof of proposition 2 - The Brownian case

- Let's first recall the scheme of the proof in the standard (Brownian noise) case.
- Setting $\mathscr{E}(t) = \sup_{s\in[0,t]} \|X_s - \widetilde{X}_s\|_{L^p(\Omega)}$, we may use Minkoswki integral inequality and BDG inequality to obtain the upper-bound

$$\mathscr{E}(t) \preccurlyeq \int_0^t \|b(X_s) - b(\widetilde{X}_{\eta(s)})\|_{L^p(\Omega)} ds + \left(\int_0^t \|\sigma(X_s) - \sigma(\widetilde{X}_{\eta(s)})\|^2_{L^p(\Omega)} ds\right)^{\frac{1}{2}}.$$

- Then we use the Lipschitz property of b and σ to get

$$\mathscr{E}(t) \preccurlyeq \int_0^t \|X_s - \widetilde{X}_{\eta(s)}\|_{L^p(\Omega)} ds + \left(\int_0^t \|X_s - \widetilde{X}_{\eta(s)}\|^2_{L^p(\Omega)} ds\right)^{\frac{1}{2}}.$$

- We may use the pivot $X_{\eta(s)}$ to separate the two terms of the right-hand side into a local error term and a Gronwall term, as follows:

$$\mathscr{E}(t) \preccurlyeq \int_0^t (\|X_s - X_{\eta(s)}\|_{L^p(\Omega)} + \mathscr{E}(s)) ds + \left(\int_0^t (\|X_s - X_{\eta(s)}\|^2_{L^p(\Omega)} + \mathscr{E}(s)^2) ds\right)^{\frac{1}{2}}.$$

- Finally we may bound the local error terms using the same inequalities and $s - \eta(s) \leq \frac{1}{n}$, allowing to apply a Gronwall-type lemma. This gives a rate of convergence of $n^{-\frac{1}{2}}$, where the power 1/2 comes from the BDG inequality.

Proof of proposition 2 - The Poisson case

- We now move to the (sketch of the) proof in our case.
- Setting $\mathscr{E}(t) = \sup_{s\in[0,t]} \|X_s - \widetilde{X}_s\|_{L^p(\Omega)}$, we may use Minkoswki integral inequality and **Kunita** inequality to obtain the upper-bound

$$\mathscr{E}(t) \preccurlyeq \int_0^t \|b(X_s) - b(\widetilde{X}_{\eta(s)})\|_{L^p(\Omega)} ds + \left(\int_0^t \mathbb{E}\left[\left(\int_{-\infty}^{\infty} (c(s, X_s, z) - c(s, \widetilde{X}_{\eta(s)}, z))^2 \nu_s(dz) \right)^{\frac{p}{2}} \right] ds \right)^{\frac{1}{p}}$$

$$+ \left(\int_0^t \int_{-\infty}^{\infty} \mathbb{E}\left[|c(s, X_s, z) - c(s, \widetilde{X}_{\eta(s)}, z)|^p \right] \nu_s(dz) ds \right)^{\frac{1}{p}}.$$

- Then we use the Lipschitz property of b and c to get

$$\mathscr{E}(t) \preccurlyeq \int_0^t \|X_s - \widetilde{X}_{\eta(s)}\|_{L^p(\Omega)} ds + \left(\int_0^t \psi_p(s) \left\| X_s - \widetilde{X}_{\eta(s)} \right\|^p_{L^p(\Omega)} ds \right)^{\frac{1}{p}},$$

where we recall that:

$$\psi_p(t) = \left(\int_{-\infty}^{\infty} |L_c(t,z) \vee |c(t,0,z)||^2 \nu_t(dz) \right)^{p/2} + \int_{-\infty}^{\infty} |L_c(t,z) \vee |c(t,0,z)||^p \nu_t(dz)$$

Sketch of the proof of Theorem 1. To prove Theorem 1, we need to do three things.

1. **Construct** a version $\widehat{X}^\varepsilon$ of the scheme that lives on the same probability space as X and satisfy an **optimal coupling** property with respect to the $\mathcal{W}_p$-distance: we will detail this construction on the next slide.

2. Find a **uniform bound** in ε for $\widehat{X}^\varepsilon$: this is done by a standard Gronwall argument using some discrete martingale properties of the construction of $\widehat{X}^\varepsilon$ and the discrete BDG inequality.

3. Derive an **upper-bound** for the L^p-norm of $\widetilde{X} - \widehat{X}^\varepsilon$: this is also done by a Gronwall argument using the optimal coupling property and Proposition 1 to get an upper bound for the small jump approximation part, leading to the contribution $\delta_p^n(\varepsilon)$ in the result of Theorem 1.

More details on the construction of $\widehat{X}^{\varepsilon}$

- For fixed x, we denote $Y_i(x) = \int_{t_{i-1}}^{t_i} \int_{B(\varepsilon)} c(s,x,z)\widetilde{N}(ds,dz)$. Our aim consists in constructing a map $T_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that **optimally transports** (for the $\mathcal{W}_p$-distance) Law$(Y_i(x))$ to the **centred normal distribution** $\mathcal{N}_i(x)$ of variance $\int_{t_{i-1}}^{t_i} \int_{B(\varepsilon)} c(s,x,z)^2 \nu_s(dz)ds$. In addition, this map must be **measurable with respect to the parameter** x.

More details on the construction of $\widehat{X}^\varepsilon$

- For fixed x, we denote $Y_i(x) = \int_{t_{i-1}}^{t_i}\int_{B(\varepsilon)} c(s,x,z)\widetilde{N}(ds,dz)$. Our aim consists in constructing a map $T_i : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ that **optimally transports** (for the $\mathscr{W}_p$-distance) Law$(Y_i(x))$ to the **centred normal distribution** $\mathcal{N}_i(x)$ of variance $\int_{t_{i-1}}^{t_i}\int_{B(\varepsilon)} c(s,x,z)^2\nu_s(dz)ds$. In addition, this map must be **measurable with respect to the parameter** x.

- We set $\widehat{X}^\varepsilon_{t_0} = X_0$. For $i\in\{1,\ldots,n\}$, let $\mathbb{Q}_i =$ Law$(\overline{X}^\varepsilon_{t_{i-1}})$. Applying Theorem 1.1 from Fontbona-Guérin-Meleard (see [FGM10]), there exists an application $T_i : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ which is $(\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}),\mathscr{B}(\mathbb{R}))$-measurable such that for $\mathbb{Q}_i$-almost every $x\in\mathbb{R}$, one has

$$\mathbb{E}[|Y_i(x) - T_i(x,Y_i(x))|^p] = \mathscr{W}_p(\text{Law}(Y_i(x)),\mathcal{N}_i(x))^p.$$

- Then, given $\widehat{X}^\varepsilon_{t_{i-1}}$, we can set

$$\widehat{X}^\varepsilon_{t_i} = \widehat{X}^\varepsilon_{t_{i-1}} + b(\widehat{X}^\varepsilon_{t_{i-1}})(t_i - t_{i-1}) + T_i\left(\widehat{X}^\varepsilon_{t_{i-1}}, Y_i(\widehat{X}^\varepsilon_{t_{i-1}})\right) + \int_{t_{i-1}}^{t_i}\int_{\mathbb{R}\setminus B(\varepsilon)} c(s,\widehat{X}^\varepsilon_{t_{i-1}},z)\widetilde{N}(ds,dz).$$

More details on the construction of $\widehat{X}^\varepsilon$

- For fixed x, we denote $Y_i(x) = \int_{t_{i-1}}^{t_i} \int_{B(\varepsilon)} c(s,x,z)\widetilde{N}(ds,dz)$. Our aim consists in constructing a map $T_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that **optimally transports** (for the $\mathcal{W}_p$-distance) Law$(Y_i(x))$ to the **centred normal distribution** $\mathcal{N}_i(x)$ of variance $\int_{t_{i-1}}^{t_i} \int_{B(\varepsilon)} c(s,x,z)^2 \nu_s(dz)ds$. In addition, this map must be **measurable with respect to the parameter** x.
- We set $\widehat{X}_{t_0}^\varepsilon = X_0$. For $i \in \{1,\dots,n\}$, let $\mathbb{Q}_i =$ Law$(\overline{X}_{t_{i-1}}^\varepsilon)$. Applying Theorem 1.1 from Fontbona-Guérin-Meleard (see [FGM10]), there exists an application $T_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ which is $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}), \mathcal{B}(\mathbb{R}))$-measurable such that for $\mathbb{Q}_i$-almost every $x \in \mathbb{R}$, one has

$$\mathbb{E}[|Y_i(x) - T_i(x, Y_i(x))|^p] = \mathcal{W}_p(\text{Law}(Y_i(x)), \mathcal{N}_i(x))^p.$$

- Then, given $\widehat{X}_{t_{i-1}}^\varepsilon$, we can set

$$\widehat{X}_{t_i}^\varepsilon = \widehat{X}_{t_{i-1}}^\varepsilon + b(\widehat{X}_{t_{i-1}}^\varepsilon)(t_i - t_{i-1}) + T_i\left(\widehat{X}_{t_{i-1}}^\varepsilon, Y_i(\widehat{X}_{t_{i-1}}^\varepsilon)\right) + \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}\setminus B(\varepsilon)} c(s, \widehat{X}_{t_{i-1}}^\varepsilon, z)\widetilde{N}(ds,dz).$$

- By construction, $\overline{X}^\varepsilon$ and $\widehat{X}^\varepsilon$ have the **same law**, and one can check that $(\widehat{X}_{t_i}^\varepsilon, 0 \le i \le n)$ is an **adapted sequence** to the filtration $(\mathcal{F}_{t_i}, 0 \le i \le n)$, and that $(T_i(\widehat{X}_{t_{i-1}}^\varepsilon, Y_i(\widehat{X}_{t_{i-1}}^\varepsilon)), 1 \le i \le n)$ is a sequence of (discrete) **martingale increments** relatively to this filtration.

There are two limitations to a numerical evaluation of the strong convergence rate in our case that we want to point out:

1. The **lack of exact trajectory solution**.

   Processing the computation of the strong norm of the error always poses the problem of simulating a reference solution trajectory, that is not available in the jump case. We chosed to compute an **approximate reference solution**, by pushing the approximation parameters to a limit value which serves as a bound for the experiments with coarse parameters. This forces us to restrict the numerical test in the increment case $c(s,x,z) = \sigma(x)f(s,z)$.

There are two limitations to a numerical evaluation of the strong convergence rate in our case that we want to point out:

1. The **lack of exact trajectory solution**.

   Processing the computation of the strong norm of the error always poses the problem of simulating a reference solution trajectory, that is not available in the jump case. We chosed to compute an **approximate reference solution**, by pushing the approximation parameters to a limit value which serves as a bound for the experiments with coarse parameters. This forces us to restrict the numerical test in the increment case $c(s,x,z) = \sigma(x)f(s,z)$.

2. The sampling of the **two-parameters increments**.

   For the ε-EM algorithm, we have two control-parameters, the time step $1/n$ and the small jumps cut ε. Once the choice of ε is **fixed**, the increments of the process $\int_0^{\cdot}\int_{\mathbb{R}/B(\varepsilon)} z\widetilde{N}(ds,dz)$ can then be simulated on a very fine time grid, and next aggregated together to produce increments on a coarser time grid.

Rate of convergence in terms of the norm exponent p

We investigate the behaviour of the L^p-strong error with respect to the variations of $p \geq 2$. For that purpose, we consider the following example:

$$X_t = \int_0^t \cos(X_s)ds + \int_0^t \int_{-\infty}^{\infty} \sin(X_{s-})\, z\, \widetilde{N}(ds,dz), \quad \nu_s(dz)ds = \mathbb{1}_{\{|z|\leq 10\}} \frac{dz}{|z|^{3/2}}\, ds.$$

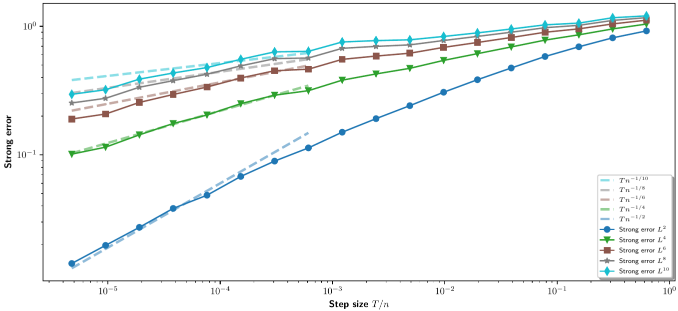


Figure 1: *Behaviour of* $\left\| \sup_{i\in\{0,\dots,n\}} \left| \overline{X}_{t_i}^{\varepsilon_{\min},n} - \overline{X}_{t_i}^{\varepsilon_{\min},n_{\max}} \right| \right\|_{L^p(\Omega)}$ *with n, for various L^p-norms (lines with makers), and the corresponding theoretical (dash lines) rates.*

Rate of convergence with low time-integrability

When ψ_p is not more than $L^{1+\zeta}$ with $\zeta \in (0,1)$, we may only recover the rate $n^{-\frac{2\zeta}{p(1+\zeta)}}$. We investigated if this loss could be observed numerically with the following equation:

$$X_t = \int_0^t \sin(X_s)ds + \int_0^t \int_{-\infty}^{\infty} \cos(X_{s^-})\, z\widetilde{N}(ds,dz), \quad \nu_s(dz)ds = \mathbb{1}_{\{|z|\leq 10\}} \frac{dz}{|z|^{3/2}}\, s^{\beta} ds, \quad \beta \in (-1,0]$$

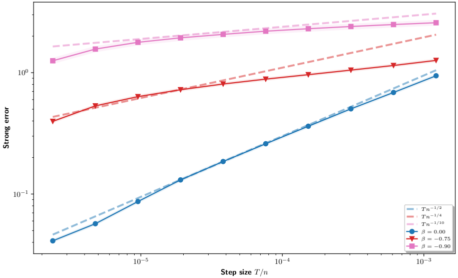


Figure 1: *Behaviour of* $\left\| \sup_{i\in\{0,\dots,n\}} \left| \overline{X}_{t_i}^{\varepsilon_{\min},n} - \overline{X}_{t_i}^{\varepsilon_{\min},n_{\max}} \right| \right\|_{L^p(\Omega)}$ *with* n*, for various* L^p*-norms (lines with makers), and the corresponding theoretical (dash lines) rates.*

9. Conclusion

- We developed a **numerical scheme** to approximate a class of **time-inhomogeneous jump SDEs** based on the **Asmussen-Rosinski** technique, and derived a rate of convergence for the L^p**-strong error** by optimal transport technique, using bounds for the CLT convergence in $\mathcal{W}_p$**-distance**.
- In the setting we are and assuming more regularity on the coefficients, we believe that it is possible to also derive a **weak error** rate for this scheme. This work is still in progress, but we have good confidence to obtain the following result:

Theorem (Work in progress)

Assume **(H1)** *and "good enough" space and time regularity of the coefficients. Assume that the ν_t are dominated by a Lévy measure μ. Let $\beta = \inf\{\alpha > 0, \int_{|z|<1} |z|^\alpha \mu(dz) < \infty\}$ the Blumenthal-Getoor index of the indivisible distribution characterised by μ.*
Let $\varphi \in \mathscr{C}^4(\mathbb{R})$ be such that for every $k \in \{1, \ldots, 4\}$,

$$\left| \frac{\partial^k \varphi}{\partial x^k}(x) \right| \leq C(1 + |x|^q) \tag{2}$$

for some $q \leq \frac{p}{2}$. We have the following weak error upper-bound:

$$|\mathscr{E}[\varphi(X_T^\varepsilon)] - \mathscr{E}[\varphi(\overline{X}_T^\varepsilon)]| \preccurlyeq n^{-1} + \varepsilon^{3-\beta^+}.$$

Victor Burte, Melina Cointe, Guy Perez, Ludovic Mailleret, and Vincent Calcagno.
When complex movement yields simple dispersal: behavioural heterogeneity, spatial spread and parasitism in groups of micro-wasps.
Movement Ecology, 11(1):13, 2023.

Sergey G. Bobkov.
Berry–Esseen bounds and Edgeworth expansions in the central limit theorem for transport distances.
Probability Theory and Relative Fields, 170(3):229 – 262, 2018.

Lorenzo Campana, Mireille Bossy, and Jérémie Bec.
Stochastic model for the alignment and tumbling of rigid fibers in two-dimensional turbulent shear flow.
Phys. Rev. Fluids, 7:124605, Dec 2022.

Rama Cont and Peter Tankov.
Financial modelling with jump processes.
Taylor & Francis eBooks DRM Free Collection, 2004.

Angelos Dassios, Jia Lim, and Yan Qu.
Exact Simulation of Truncated Lévy Subordinator.
ACM Transactions on Modeling and Computer Simulation, 30, 10 2019.

Joaquin Fontbona, Hélène Guérin, and Sylvie Méléard.
Measurability of optimal transportation and strong coupling of martingale measures.
Electronic Communications in Probability, 15(none):124 – 133, 2010.