Markovian approximation of a Volterra SDE model for intermittency

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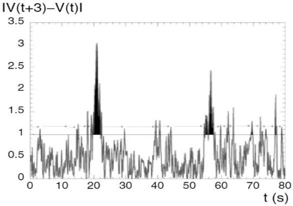
[1] This is an ongoing work, jointly with Mireille Bossy and Kerlyns Martínez.

Introduction

Wind gusts are small-scale wind fluctuations that are by nature **intermittent**.

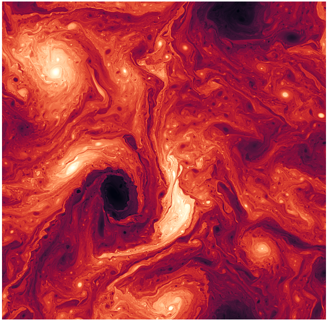


Given a time scale τ (here 3 seconds) and a threshold δ (here 1 m/s), characterising intermittent fluctuation $\|\Delta U(\tau)\| = \|U(t+\tau) - U(t)\| > \delta$ having **non Gaussian** properties

- Some predictive frameworks are ready to use, but assuming **Gaussian statistics**.
- Goal : develop a **stochastic model** that take into account **Kolomogorov's refined theory**. This involves stochastic processes with **memory**.
- Kolmogorov's theory predicts **multiscaling** such as anomalous power-laws emerging at the level of the velocity increments : $\mathbb{E}[|\Delta U(\tau)|^p] \simeq \tau^{\zeta(p)}$, with ζ **non-linear** function.

Plan of the talk

1. **Physical context** and **modelling**
2. A **Volterra** process and its **Markovian approximation**
3. A **martingale** approach based on **orthogonal decomposition**
4. **Weak convergence** analysis of the Markovian approximation



A direct numerical simulation of 2D turbulence, provided by Nicolas Valade (Calisto Team INRIA)

- **Navier-Stokes** equation: $\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = -\nabla p + \frac{1}{\mathrm{Re}} \Delta \vec{u}$
- **Energy dissipation**: $\varepsilon(t,x) = \frac{\nu}{2} \langle \mathrm{trace} \nabla^T u \nabla u) \rangle (t,x)$
- In **Lagrangian** setting,

$$X_t = X_0 + \int_0^t u(s, X_s) ds$$
$$\varepsilon_t = \frac{\nu}{2} \langle \mathrm{trace} \nabla^T u \nabla u)(t, X_t) \rangle$$

Kolmogorov's refined theory for fluctuations of the **energy dissipation** ε (can be seen as the volatility behind the velocity U): [Kolmogorov, 1962] [Frisch and Parisi, 1985] [Frisch, 1995]:

- **stationarity** and **scaling** : $\mathbb{E}[\varepsilon_t] = \nu \tau_\eta^{-2}$ (Kolmogorov 1941);
- **log-normality** of ε: with $\mathbb{V}\mathrm{ar}[\log \varepsilon_t] \simeq \log\left(\frac{\tau_L}{\tau_\eta}\right)$; $\qquad \tau_L = \frac{1}{\langle \|u\|^2 \rangle(t)} \int_0^{+\infty} \langle u(t+\theta)u(t)\rangle d\theta$
- **multiscaling** of the one-point statistics: $\mathbb{E}[\varepsilon_t^p] \simeq \left(\frac{T_L}{\tau_\eta}\right)^{\zeta(p)}$, where $\zeta(p)$ is a non-linear convex function;
- **power-law scaling** for the **coarse-grained** dissipation and the velocity: in the inertial range, $\tau_\eta \ll \tau \ll T_L$,

$$\mathbb{E}\left[\left|\frac{1}{\tau}\int_t^{t+\tau} \varepsilon_s\right|^p\right] \simeq \tau^{\zeta(p)},$$
$$\mathbb{E}[|U(t+\tau) - U(t)|^p] \simeq \tau^{\zeta(p)}.$$

We seek to construct a stationary process $\varepsilon_t = \overline{\varepsilon}\exp(\gamma X_t - \frac{\gamma^2}{2}\operatorname{Var} X_0)$, where X is a **log-correlated** stationary process.

[Forde et al., 2022] consider the re-scaled **Riemann–Liouville** fractional Brownian motion (Z_t^H),

$$Z_t^H = \int_0^t (t-s)^{-\frac{1}{2}+H} dW_s$$

for $H \in (0, \frac{1}{2})$, for which $R_H(s,t) := \mathbb{E}[Z_s^H Z_t^H] = \int_0^{s\wedge t}(s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}} du$. Also, $R_H(s,t) \to R(s,t)$, with $H \to 0$, and for $s < t$,

$$R(s,t) = \ln\left(\frac{(\sqrt{t}+\sqrt{s})^2}{t-s}\right) = \ln(\frac{1}{t-s}) + 2\ln(\sqrt{t}+\sqrt{s}).$$

They showed that $\phi_\gamma^H(dt) = \exp\left(\gamma Z_t^H - \frac{\gamma^2}{2}\operatorname{Var}[Z_t^H]\right) dt$ tends to a **Gaussian multiplicative chaos** (GMC) random measure ϕ_γ, for $\gamma \in (0,1)$, as H tends to zero. (the convergence is in law for $\gamma \in (0, \sqrt{2})$).

- [Letournel, 2022] propose, in her PhD thesis, the following **doubly regularised** H-fBm as a stationary process.
- Consider, for $0 < \tau_\eta < \tau_L$,

$$X_t^{H,\tau_\eta,\tau_L} = \int_{-\infty}^{t} \left[(t-r+\tau_\eta)^{H-\frac{1}{2}} - (t-r+\tau_L)^{H-\frac{1}{2}} \right] dW_r \tag{1}$$

- For $H = 0$, the process is still **well defined** and **stationary**.
- One can compute its correlation function and variance, and

$$R^{0,\tau_\eta,\tau_L}(s,t) = \mathbb{E}[X_t^{0,\tau_\eta,\tau_L} X_s^{0,\tau_\eta,\tau_L}] = R(t-s) \sim \log_+ \frac{1}{t-s}, \quad \tau_\eta \ll s < t \ll \tau_L$$

$$\text{Var}[X_t^{0,\tau_\eta,\tau_L}] \sim \log(\tau_L/\tau_\eta)$$

1. Physical context : a stochastic model for the velocity

- Based on this stationary Volterra process, we are able to construct a **stochastic model** for the dissipation and velocity:

$$
\begin{aligned}
\varepsilon_t =& \overline{\varepsilon} \exp\left(\gamma X_t^{0,\tau_\eta,\tau_L} - \frac{\gamma^2}{2}\mathrm{Var}(X_0^{0,\tau_\eta,\tau_L})\right) \; ; \qquad \gamma, \overline{\varepsilon} > 0, \\
U_t =& U_0 - \int_0^t \frac{1}{\tau_L} U_s ds + \int_0^t \sqrt{\varepsilon_s} dB_s,
\end{aligned}
$$

  where B is a standard Brownian motion **independent** from W.

- We arbe interested in the statistics of the **coarse-grained dissipation**:

$$
D_t^{t_0} = \frac{1}{t}\int_{t_0}^{t_0+t} \varepsilon_s ds \; ; \qquad t, t_0 \geq 0.
$$

  and of the velocity increments

$$
U_t^{t_0} = U_{t+t_0} - U_{t_0} \; ; \qquad t, t_0 \geq 0.
$$

- We will say that U^{t_0} and D^{t_0} are **integrated models** with respect to the underlying Volterra process X^{0,τ_η,τ_L}.

2. A Volterra process. Mathematical setting

Fix $T > 0$ and denote $\mathbb{T} = [0, T]$. We consider:

- $W = (W_t^-, W_t^+)_{t \in \mathbb{R}_+}$ a standard **2D Brownian Motion**,
- $K : \mathbb{R}_+ \to \mathbb{R}$ be a **completely monotone kernel** : $K(r) = \int_0^{+\infty} e^{-rt} \lambda(x) dx, \quad \lambda \nearrow$
- F the unique primitive of K^2 such that $F(0) = 0$.

Define:

$$\forall t \in \mathbb{T} \quad X_t = \int_{-\infty}^{t} K(t-s) dW_s$$
$$= \underbrace{\int_0^{+\infty} K(t+s) dW_s^-}_{X_t^-} + \underbrace{\int_0^{t} K(t-s) dW_s^+}_{X_t^+}.$$

Proposition

Assume that $F(\infty) = \lim_{x \to +\infty} F(x)$ exists and lies in $(0, +\infty)$. Then the process X is a well-defined **stationnary Gaussian process**, with covariance function

$$\forall (s,t) \in \mathbb{T}^2, \quad \mathbb{E}[X_s X_t] \quad = \quad \int_{-\infty}^{t \wedge s} K(t-r) K(s-r) dr,$$

and variance $\forall t \in \mathbb{T}, \mathrm{Var}(X_t) = F(\infty)$.

The process X^+ (and X when it makes sense) is:

- A **semimartingale** with respect to the filtration of W^+ **if and only if** $\int_0^T \left(\frac{\partial K}{\partial r}(r)\right)^2 dr < \infty$ (from [Basse, 2009][Theorem 4.6]).
- **Not a Markov process** except if K is constant.

Example

- For the H-fractional kernel $K(r) = r^{H-1/2}$, X^+ is **non-Markov** and **non-semimartingale**.
- For $K(r) = (r+\tau_\eta)^{-1/2} - (r+\tau_L)^{-1/2}$, X and X^+ are **non-Markov semimartingales**.

The processes X^+ and X belong to the wider class of **Stochastic Volterra Equations**:

$$X_t = x_t + \int_0^t b(t,s,X_s)ds + \int_0^t \sigma(t,s,X_s)dW_s^+,$$

with $b = 0$, $\sigma(t,s,x) = K(t-s)$, and respectively $x_t = 0$ and $x_t = X_t^-$.

Let ϕ be a smooth real-valued function.

- Computing $\mathbb{E}[\phi(X_T)]$ by Monte-Carlo method is not a big deal since the law of X_T is **Gaussian** and X_T can be sampled exactly.
- However, due to the **non-Markovianity**, there is no systematic way to generate a **trajectory** of X until time T.
- This is required to estimate statistics of **integrated models** based on X such as

$$\mathbb{E}\left[\phi\left(\int_0^T \psi(s, X_s)ds\right)\right].$$

Let $0 = t_0 < t_1 < \cdots < t_n = T$ be a **discretisation** of $\mathbb{T}$. We are interested in the **approximation of a trajectory** $(X_{t_0}, \ldots, X_{t_n})$ of the process X and associated **convergence analysis**.

Applying the **stochastic Fubini theorem** and discretising the **Laplace transform** of K, we get:

$$
\begin{aligned}
\int_0^t K(t-s)dW_s^+ &= \int_0^t \left(\int_0^{+\infty} e^{-(t-s)x}\lambda(x)dx\right) dW_s^+ \\
&= \int_0^{+\infty} \lambda(x)dx \left(\int_0^t e^{-(t-s)x} dW_s^+\right) \\
&\simeq \sum_{i=1}^m w_i Y_t^{x_i},
\end{aligned}
$$

where

- $(w_i, x_i)_{\{1\leq i\leq m\}}$ is an appropriate **Gauss quadrature** of order m for $\int_0^{+\infty} f(x)\lambda(x)dx$,
- $(Y_t^{x_i})_{t\in[0,T]}$ is a (**Markov**) **Ornstein-Uhlenbeck** process starting from zero :

$$
\begin{aligned}
dY_t^{x_i} &= -x_i Y_t^{x_i} dt + dW_t^+ \\
Y_0^{x_i} &= 0.
\end{aligned}
$$

This formal discussion suggests the **following strategy** to approximate $(X_{t_0}, \dots, X_{t_n})$.

1. Sample the "initial condition" $(X^-_{t_0}, \dots, X^-_{t_n})$.
2. Sample the **correlated** OU processes $(Y_t^{x_1}, \dots, Y_t^{x_m})_{t \in \{t_0, \dots, t_n\}}$.
3. Compute $X_{t_i} = X^-_{t_i} + \sum_{i=1}^m w_i Y_{t_i}^{x_i}$ for each $i \in \{t_0, \dots, t_n\}$.

Example

For $K(r) = (r + \tau_\eta)^{-1/2} - (r + \tau_L)^{-1/2}$, one has the Markovian representation

$$X_t = X_t^- + \int_0^{+\infty} \frac{e^{-\tau_\eta x} - e^{-\tau_L x}}{\sqrt{x}} \left(\int_0^t e^{-(t-s)x} dW_s^+ \right) dx.$$

Pertinent choices for the quadrature are:

- Fix an upper-bound $B > 0$ and use Gauss-Jacobi weights and nodes for integrals of the form $\int_{-1}^1 f(x)(1-x)^\alpha (1+x)^\beta dx$ to approximate $\int_0^B \lambda(x) Y_t^x dx$ (with $\alpha = 0; \beta = -0.5$).
- Use weights and nodes associated to generalised Gauss-Laguerre quadrature for integrals of the form $\int_0^{+\infty} f(x) x^\alpha e^{-x} dx$ with $\alpha = -0.5$.

- For $m \in \mathbb{N}^*$, we set $K_m(r) = \sum_{i=1}^m w_i e^{-x_i r}$, so that

$$\sum_{i=1}^m w_i Y_t^{x_i} = \int_0^t K_m(t-s) dW_s^+,$$

- We set for $t \in \mathbb{T}$:

$$X_t^m := X_t^- + X_t^{+,m} = X_t^- + \int_0^t K_m(t-s) dW_s^+.$$

One has the following **strong convergence** rate from [Alfonsi and Kebaier, 2024]:

Theorem (Theorem 3.1 from [Alfonsi and Kebaier, 2024])

There exists a constant $C \geq 0$ such that for any $t \in \mathbb{T}$,

$$\mathbb{E}[|X_t^+ - X_t^{+,m}|^2] \leq C \left(\int_0^t |K(s) - K_m(s)|^2 ds \right).$$

We are interested in the **weak convergence rate**, i.e in finding an upper-bound for

$$|\mathbb{E}[\phi(X_T)] - \mathbb{E}[\phi(X_T^m)]|$$

in terms of a distance between K and K^m, where ϕ is a smooth real-valued function.

Let $\mathscr{F}_t = \sigma(W^+_s \; ; \; s \in [0,t])$ for $t \in \mathbb{T}$.

- Standard method of proof for weak convergence relies on the **regularity** of $u(t,x) = \mathbb{E}[\phi(X_T)|X_t = x]$ and the associated **PDE**.
- This does not apply to X which is a non-Markovian process. However, one can dismantle X_t with a Chasles relation:

$$\begin{aligned} X_s &= X^-_s + \Theta^t_s + I^t_s \\ &= \int_0^{+\infty} K(s+r)dW^-_r + \int_0^t K(s-r)dW^+_r + \int_t^s K(s-r)dW^+_r, \end{aligned}$$

where:

- $\Theta^t_s = \int_0^t K(s-r)dW^+_r$ is $\mathscr{F}_t$-**measurable** for all $t \leq s$,
- $X^-_s = \int_0^{+\infty} K(s+r)dW^-_r$ and $I^t_s = \int_t^s K(s-r)dW^+_r$ are **independent** of $\mathscr{F}_t$.

Note that in particular, $(\Theta^t_T)_{t\in\mathbb{T}}$ is a $\mathscr{F}$-**martingale**.

Using the **orthogonal decomposition** presented above and the usual properties of conditional expectation, we have for all $t \in \mathbb{T}$:

$$\begin{aligned} \mathbb{E}[\phi(X_T)|\mathcal{F}_t] &= \mathbb{E}[\phi(X_T^- + \Theta_T^t + I_T^t)|\mathcal{F}_t] \\ &= u(t, \Theta_T^t), \end{aligned}$$

where $u(t,x) := \mathbb{E}[\phi(X_T^{t,x})] = \mathbb{E}\left[\phi\left(X_T^- + x + I_T^t\right)\right]$.

By **Gaussian computations**, one easily show the following Lemma:

Lemma

For $x \in \mathbb{R}$ and $(s,t) \in \mathbb{T}^2$ such that $t \leq s$, let

$$X_s^{t,x} = X_s^- + x + I_s^t.$$

Then for any $p \in \mathbb{R}_+$, there exists a constant $C_p \in (0, +\infty)$ such that

$$\sup_{s \in [t,T]} \mathbb{E}[e^{pX_s^{t,x}}] \leq C_p e^{px}.$$

We make the following assumption on the regularity of ϕ:

Hypothesis (H1)

$\phi \in \mathscr{C}^3(\mathbb{R})$ and that there exists $C_\phi, \kappa_\phi > 0$ such that for $g \in \{\phi, \phi', \phi''\}$,

$$g(x) \leq C_\phi(1 + e^{\kappa_\phi x}).$$

This lead us to the following result on the regularity of u:

Proposition

Assume **(H1)**. Then $u \in \mathscr{C}^{1,2}(\mathbb{R})$ and

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}K^2(T-t)\frac{\partial^2 u}{\partial x^2}(t,x) \\ u(T,x) &= \mathbb{E}[\phi(X_T^- + x)]. \end{aligned}$$

3. The orthogonal decomposition : a PDE satisfied by u

Scheme of proof:

- Using the **regularity** of ϕ, one can show that $u(t,\cdot) \in \mathscr{C}^2(\mathbb{R})$ by theorem of differentiation under the sign $\mathbb{E}$, and then that $u(\cdot,x)$ is **absolutely continuous** applying Itô formula on $\phi(X_T^- + x + I_T^s)$ between $s = t+h$ and $s = t$.

- Applying the **Itô formula** on $u(t,\Theta_T^t)$, we obtain

$$du(t,\Theta_T^t) = \left(\frac{\partial u}{\partial t}(t,\Theta_T^t) + \frac{1}{2}K^2(T-t)\frac{\partial^2 u}{\partial x^2}(t,\Theta_T^t)\right)dt + K(T-t)\frac{\partial u}{\partial x}(t,\Theta_T^t)dW_t^+.$$

- But $u(t,\Theta_T^t) = \mathbb{E}[\phi(X_T)|\mathscr{F}_t]$ is a **martingale**, hence **the drift term must vanish**, that is

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) + \frac{1}{2}K^2(T-t)\frac{\partial^2 u}{\partial x^2}(t,x) &= 0 \\ u(T,x) &= \phi(x). \end{aligned}$$

This leads us to the following weak convergence result for X_t^m:

Proposition

Assume **(H1)**. Then there exists a constant $C > 0$ independent of m such that:

$$|\mathbb{E}[\phi(X_T)] - \mathbb{E}[\phi(X_T^m)]| \quad \leq \quad C \left| \int_0^T K^2(r) - K_m^2(r) dr \right|.$$

Note that for $\phi(x) = x^2$, the Itô isometry yields

$$|\mathbb{E}[\phi(X_T)] - \mathbb{E}[\phi(X_T^m)]| = \left| \int_0^T (K^2(r) - K_m^2(r)) dr \right|.$$

Scheme of proof:

- From the proposition above one can write $|\mathbb{E}[\phi(X_T)] - \mathbb{E}[\phi(X_T^m)]| = |\mathbb{E}[u(T, X_T^m) - u(0,0)]|$.
- Set $\Theta_s^{t,m} = \int_0^t K_m(s-r)dW_r^+$ and applying **Itô formula**:

$$\mathbb{E}[u(T,\Theta_T^{T,m}) - u(0,0)] = \mathbb{E}\int_0^T \left\{ \frac{\partial u}{\partial t}(s,\Theta_T^{s,m}) + \frac{1}{2}K_m^2(T-s)\frac{\partial^2 u}{\partial x^2}(s,\Theta_T^{s,m}) \right\} ds.$$

- Using the **PDE** satisfied by u, the right hand-side boils down to

$$\frac{1}{2}\mathbb{E}\int_0^T \left\{ \frac{\partial^2 u}{\partial x^2}(s,\Theta_T^{s,m})(K_m^2(T-s) - K^2(T-s)) \right\} ds.$$

- Then an easy way to conclude would be to push the **absolute value inside** and use the upper-bound

$$\mathbb{E}\left|\frac{\partial^2 u}{\partial x^2}(s,\Theta_T^{s,m})\right| \leq C_{\phi''}\left(1 + \sup_{s\in\mathbb{T}}\mathbb{E}\left[e^{\kappa_{\phi''}\, X_T^{s,\Theta_T^{s,m}}}\right]\right) < \infty,$$

  but this will lead to a worse rate of convergence.
- Instead, we use a development to the **second order** by applying Itô formula again to $\frac{\partial^2 u}{\partial x^2}(s,\Theta_T^{s,m})$.

- Consider $D_t = \int_0^t \psi(X_s)ds$ with ψ being a positive, smooth function. One has the less trivial **martingale decomposition**:

$$\begin{aligned}\mathbb{E}[\Phi(D_T)|\mathscr{F}_t] =& \mathbb{E}\left[\Phi\left(D_t + \int_t^T \psi^2(X_s)ds\right)|\mathscr{F}_t\right] \\ =& \mathbb{E}\left[\Phi\left(D_t + \int_t^T \psi^2(\Theta_s^t + I_s^t)ds\right)|\mathscr{F}_t\right] \\ =& v(t, D_t, \Theta^t_{[t,T]}),\end{aligned}$$

where $v(t,x,\omega) = \mathbb{E}_{t,x,\omega}[\Phi(D_T)] = \mathbb{E}[\Phi(D_T)|D_t = x, \Theta^t = \omega]$ for all $(t,x,\omega) \in \mathbb{T} \times \mathbb{R} \times C(\mathbb{T},\mathbb{R})$.

- One can show that v satisfies the following **Path-Dependent PDE** (PPDE):

$$\begin{aligned}\frac{\partial v}{\partial t}(t,x,\omega) + \psi^2(\omega_t)\frac{\partial v}{\partial x}(t,x,\omega) + \frac{1}{2}\psi^2(\omega_t)\frac{\partial^2 v}{\partial x^2}(t,x,\omega) + \frac{1}{2}\left\langle \frac{\partial^2 v}{\partial \omega^2}(t,x,\omega), (G^t, G^t)\right\rangle =& 0 \\ v(T,x,\omega) =& \Phi(x)\end{aligned}$$

- Using the **orthogonal decomposition** from [Viens and Zhang, 2019], we are able to obtain a **weak convergence result** for the **Markovian approximation** of $(X_t)_{t\in\mathbb{T}}$.
- We believe that the rate of convergence obtained at the level of the process $(X_t)_{t\in\mathbb{T}}$ can be **extended to integrated models**.
- This would require to deal with **functional Itô formula** and **path-dependent PDEs** (see [Viens and Zhang, 2019] and [Bonesini et al., 2023]), making the proofs more intricate.

Beyond this generalisation, it would be also interesting to:

1. Understand how the L^1-**distance** between K^2 and K_m^2 can be controlled in the case of Letournel's kernel, depending on the chosen quadrature method.
2. Integrate the **approximation of the initial condition** in the convergence analysis (for the stationary case $X_t^- \neq 0$)
3. From a modelling point of view, understand better in what extend the **intermittency** properties such as $\mathbb{E}[|\int_t^{t+\tau}\varepsilon_s ds|^p] \simeq \tau^{\zeta(p)}$ are recovered with the Volterra model and with its Markovian approximation.

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