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Discretisation scheme for time-inhomogeneous jump SDEs.

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When they are driven by a Lévy process whose increments can be simulated exactly, the simulation of stochastic differential equation with jumps can be straightforward via an Euler scheme. However, this is only possible in a few restrictive cases, which arise the need for a general method that works for a larger class of SDEs. The approach taken here follows the ideas of Asmussen-Rosinski in [AR01], and consists in simulating exactly the large jumps of the process, while approximating the small jumps by Gaussian variables. We are inspired by the ideas of Fournier in [Fou12], using Wasserstein techniques to perform a strong error analysis.

L^p-wellposedness

Our generic SDE, defined for $t \in [0,T]$:

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s + \int_0^t \int_{-\infty}^{\infty} c(s, X_{s^-}, z)\tilde{N}(ds, dz), \qquad (1)$$

- $N(ds, dz)$ is a random Poisson measure with time-dependent compensator $\nu_s(dz)ds$
- $\tilde{N}(ds, dz) = N(ds, dz) - \nu_s(dz)ds$ is the compensated Poisson measure.

L^p-wellposedness framework of [BP20] with time-Hölder condition:

(H1) - Time-Hölder and Lipschitz: The drift and diffusion coefficients $a(t,x)$ and $b(t,x)$ are Hölder w.r.t. t and Lipschitz w.r.t. x. The jump coefficient $c(t,x,z)$ verifies:

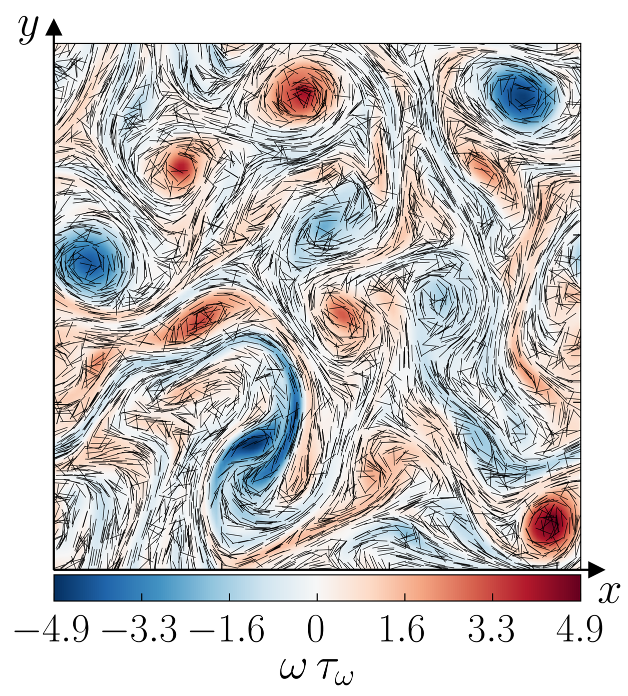
$$|c(t,x,z) - c(t,y,z)| \leq L_c(t,z)|x-y|, \quad x, y \in \mathbb{R}, \ t \in [0,T].$$

(H2) - Integrability: $X_0 \in L^p(\Omega)$ and

$$\int_0^T \left(\int_{-\infty}^{\infty} |F(t,z)|^2 \nu_t(dz) \right)^{p/2} dt < \infty, \quad \int_0^T \int_{-\infty}^{\infty} |F(t,z)|^p \nu_t(dz) < \infty,$$

for $F(\cdot,\cdot) = L_c(\cdot,\cdot)$ and $F(\cdot,\cdot) = c(\cdot, 0, \cdot)$.

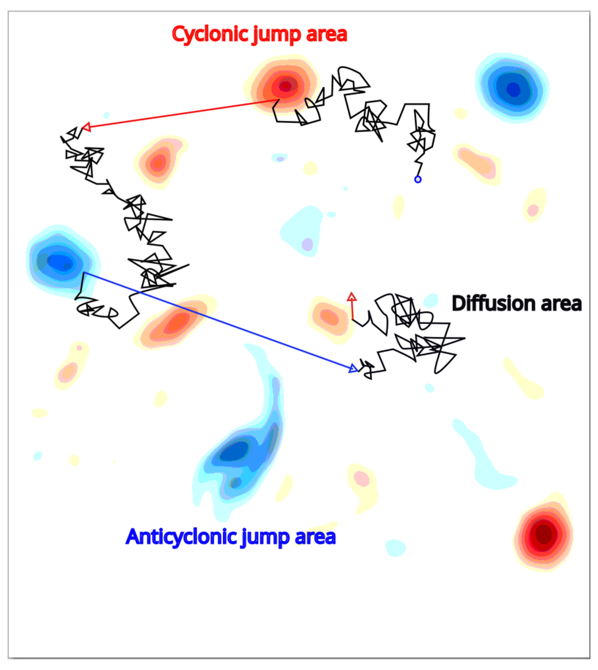
Application : stochastic orientation of rods in 2D turbulence



We consider intertialess rods in a turbulent flow with position equation $dX(t)/dt = v(X(t), t)$, coupled with a unit orientation vector p following Jeffery's equation :

$$\frac{d}{dt}p = \mathbb{A}p - (p^T \mathbb{A}p)p, \qquad (2)$$

where $\mathbb{A}$ denotes the gradient tensor of the fluid-velocity v. The figure on the left, obtained in [CBB22], shows the vorticity and velocity fields obtained as Direct Numerical Simulation (DNS) of 2D Navier Stokes.



After averaging on the gradient tensor $\mathbb{A}$ at the equilibrium regime, a Brownian SDE followed by the unfolded angle $\theta_t = \arctan(p_2/p_1)$ has been derived in [CBB22]. To take in account the effects of the vertices as brutal variations of the angular displacement, we added a jump term leading to

$$\theta_t = \int_0^t a(\theta_s)ds + \int_0^t b(\theta_s)dW_s + \int_0^t \int_{-\infty}^{\infty} b(\theta_s)\tilde{N}(ds, dz), \qquad (3)$$

where $a(x)$ and $b(x)$ being linear combining of $\cos(x)$ and $\sin(x)$, $\nu_s(dz) = (s^{1/2} \wedge T^*)|z|^{-1-\alpha} \mathbb{1}_{\{|z|<T^*\}}$, T^* being the average lifetime of the vertices.

General framework for the simulation of a Poisson integral

The stochastic Poisson integral can be separated into *large jumps* and *small jumps*: $\int_0^T \int_{-\infty}^{+\infty} F(s,z)\tilde{N}(ds,dz) = \int_0^T \int_{\mathbb{R}\backslash B(\varepsilon)} F(s,z)\tilde{N}(ds,dz) + \int_0^T \int_{B(\varepsilon)} F(s,z)\tilde{N}(ds,dz)$.

A. Numerically tractable representation of large jumps

$$\int_0^T \int_{\mathbb{R}\backslash B(\varepsilon)} F(s,z)N(ds,dz) = \sum_{j=1}^{N^\varepsilon(T)} F(T^\varepsilon(j), Z^\varepsilon(j))\mathbb{1}_{\{\mathbb{R}\backslash B(\varepsilon)\}}(Z^\varepsilon(j)), \quad \text{where} \quad \begin{cases} N^\varepsilon(T) \text{ is a time-inomogeneous Poisson process with intensity } \lambda^\varepsilon(t) = \int_{\mathbb{R}\backslash B(\varepsilon)} \nu_t(dz), \\ T^\varepsilon(j) = \inf\{t \in [0,T], N^\varepsilon(t) = j\} \text{ are the jump times of } N^\varepsilon, \\ Z^\varepsilon(j) = \int_{\mathbb{R}\backslash B(\varepsilon)} zN(T^\varepsilon(j), dz) - \int_{\mathbb{R}\backslash B(\varepsilon)} zN(T^\varepsilon(j-1), dz) \text{ are the jump sizes of } N^\varepsilon. \end{cases}$$

The jump sizes $Z^\varepsilon(j)$ has conditional distribution $\mathbb{P}(Z^\varepsilon(j) \in B \mid T^\varepsilon(j) = t) = \frac{\nu_t(B \cap \mathbb{R}\backslash B(\varepsilon))}{\nu_t(\mathbb{R}\backslash B(\varepsilon))}$ and can be simulated by inversion or rejection method for simple cases.

B. Gaussian approximation of the small jumps

We approximate the *small jumps* by a Gaussian r.v, making use of the following bound in L^p-Wasserstein distance. The proof rely on a Berry-Essen type bound for the CLT obtained by Bobkov in [Bob18].

Proposition: L^p-Wasserstein approximation bound
Assume that F satisfies **(H2)** for $p \geq 2$ and that for any $s \in [0,T]$, $F(s,\cdot)$ does not vanish on $B(\varepsilon)$ and has at most polynomial growth. Then, there exists a constant $\mathcal{A}_p$ only depending on p such that the following inequality holds for any $r, t \in [0,T]$:

$$\mathcal{W}_p\left(\mathscr{L}aw\left(\int_r^t \int_{B(\varepsilon)} F(s,z)\tilde{N}(ds,dz) \right), \mathcal{N}\left(0, \int_r^t \int_{B(\varepsilon)} |F(s,z)|^2 \nu_s(dz)ds \right) \right)^p \leq \mathcal{A}_p \frac{\int_r^t \int_{B(\varepsilon)} |F(s,z)|^{p+2} \nu_s(dz)ds}{\int_r^t \int_{B(\varepsilon)} |F(s,z)|^2 \nu_s(dz)ds} \xrightarrow[\varepsilon \to 0]{} 0$$

Numerical scheme

Let $t_i = i\frac{T}{n}$. Define the numerical scheme $\overline{X}$ by $\overline{X}_{t_0} = X_0$ and for each $i \in \{1, \ldots, n\}$:

$$\overline{X}_{t_i} = \overline{X}_{t_{i-1}} + a(t_{i-1}, \overline{X}_{t_{i-1}})\frac{T}{n} + b(t_{i-1}, \overline{X}_{t_{i-1}})(W_{t_i} - W_{t_{i-1}}) + B(\overline{X}_{t_{i-1}}) + \int_{t_{i-1}}^{t_i} \int_{\mathbb{R}\backslash B(\varepsilon)} c(s, \overline{X}_{t_{i-1}}, z)\tilde{N}(ds,dz), \qquad (4)$$

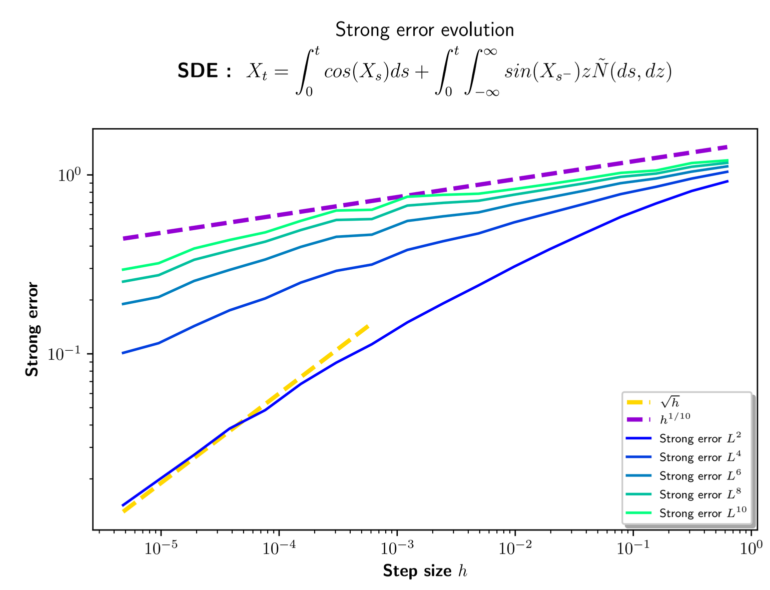
where the Gaussian r.v. $B(\overline{X}_{t_{i-1}})$ is the 2$^{\text{nd}}$ component of the L^p-Wasserstein optimal coupling between $\mathscr{L}aw\left(\int_{t_{i-1}}^{t_i} \int_{B(\varepsilon)} c(s,x,z)\tilde{N}(ds,dz) \right)$ and $\mathcal{N}\left(0, \int_{t_{i-1}}^{t_i} \int_{B(\varepsilon)} c(s,x,z)\nu_s(dz)ds \right)$, taken at $x = \overline{X}_{t_{i-1}}$.

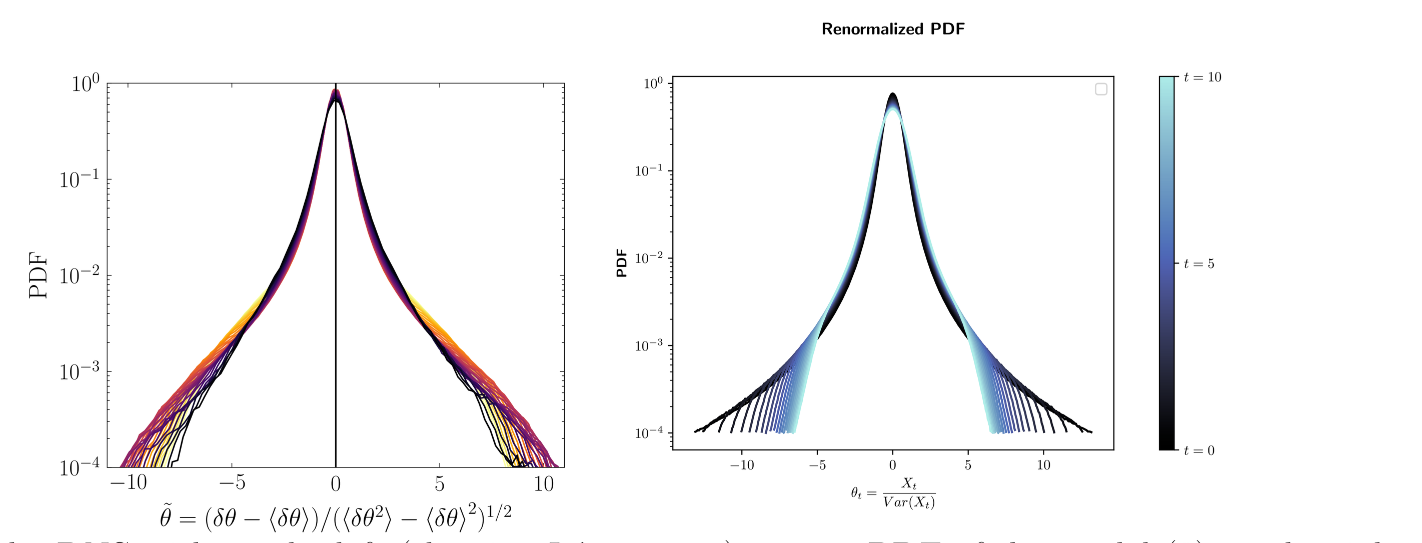
Main theorem : Strong error convergence

Assume **(H1)**, **(H2)**, and that $c(s,x,\cdot)$ does not vanish on $B(\varepsilon)$. Let $\eta(t) = t_i$ for $t \in [t_i, t_i + 1)$. Then:

1. There exists $m(p,T) > 0$ such that $\sup_{t \in [0,T]} \mathbb{E}[|\overline{X}_{\eta(t)}|^p] \leq m(p,T) < \infty$,
2. One has $\mathbb{E}\left\{ \sup_{t \in [0,T]} |X_{\eta(t)} - \overline{X}_{\eta(t)}|^p \right\} \leq C_{p,T}\left(\frac{1}{n^{(p\gamma)\wedge 1}} + \Delta_{n,p}(\varepsilon) \right)$, where $C_{p,T} > 0$ does not depends on n, and the coefficient $\Delta_{n,p}(\varepsilon)$ is defined by
$$\Delta_{n,p}(\varepsilon) = \sum_{i=1}^n \mathbb{E}\left[\frac{\int_{t_{i-1}}^{t_i} \int_{B(\varepsilon)} |c(s, \overline{X}_{t_{i-1}}, z)|^{p+2} \nu_s(dz)ds}{\int_{t_{i-1}}^{t_i} \int_{B(\varepsilon)} |c(s, \overline{X}_{t_{i-1}}, z)|^2 \nu_s(dz)ds} \right].$$
3. If in addition $c(s,x,\cdot)$ has at most polynomial growth, then ε can be chosen small enough to get:
$$\left\| \sup_{t \in [0,T]} |X_{\eta(t)} - \overline{X}_{\eta(t)}| \right\|_{L^p(\Omega)} \leq C'_{p,T} \frac{1}{n^{\gamma \wedge (1/p)}}.$$

Numerical simulations





PDF of the DNS with on the left (showing Lévy wings), versus PDF of the model (3) on the right.

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