Stochastic modelling of rigid fibres in turbulence

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We present two types of stochastic models for rigid fibres in turbulence. The first one is a Poisson-driven càdlàg SDE that describes the orientation of small rods in two-dimensional turbulence. The second one involves the derivation of an SDE model for long rigid fibres in a turbulent fluid, where the turbulence is modeled by infinite-dimensional Kraichnan noise. For the jump model, we introduce an appropriate numerical scheme and prove an optimal $L^p(\Omega)$-strong rate of convergence, as well as a weak rate of convergence. For the three-dimensional Kraichnan model, we establish the well-posedness of the equation and prove its equivalence in law to a finite-dimensional SDE.

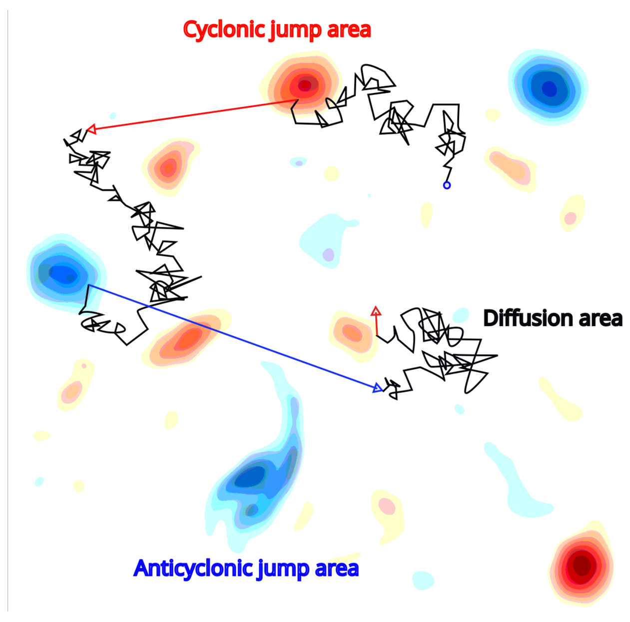
Preprint available on ArXiv (jump model)

A Poisson-driven model for orientation

We consider intertialless small rods in a turbulent flow with position equation $dX(t)/dt = v(X(t),t)$, coupled with a unit orientation vector p following Jeffery's equation :

$$\frac{d}{dt}p = \mathbb{A}p - (p^T \mathbb{A}p)p, \qquad (1)$$

After averaging on the gradient tensor $\mathbb{A}$, Brownian SDE derived in [CBB22] for the unfolded angle $\theta_t = \arctan(p_2/p_1)$.



We transform it into a Poisson-SDE for θ. For $t \in [0,T]$, set

$$d\theta_t = b(\theta_t)dt + \sigma(\theta_{t-}, z)\tilde{N}(dt, dz),$$

N is a Poisson random measure with intensity $\nu_t(dz)dt = \mathbb{E}[N(dt,dz)]$,

$$\nu_t(dz) = (t^{1/2} \wedge T^*)|z|^{-1-\alpha}\, \mathbb{1}_{\{|z|<T^*\}},$$

$\alpha \in (0,2]$, T^* vertices average lifetime, $b, \sigma(\cdot, z)$ trigonometric functions.

The ε-EM scheme for Poisson SDEs

To approximate the Poisson-SDE satisfied by θ_t we introduce, for a number of time steps $n \in \mathbb{N}\backslash\{0\}$ and a threshold $\varepsilon > 0$, the scheme $\overline{\theta}^\varepsilon$ defined by $\overline{\theta}^\varepsilon_0 = \theta_0$ and for $0 \le i \le n-1$:

$$\overline{\theta}^\varepsilon_{t_{i+1}} = (b - b_\varepsilon)(\overline{\theta}^\varepsilon_{t_i})\Delta t + \sigma_\varepsilon(\overline{\theta}^\varepsilon_{t_i})^{\frac{1}{2}}\xi_i + J(\overline{\theta}^\varepsilon_{t_i}), \quad t_i = iT/n \qquad (3)$$

In this scheme,

- $b_\varepsilon(x) = \int_{t_i}^{t_{i+1}} \int_{(-\infty,-\varepsilon]\cup[\varepsilon,+\infty)} \sigma(x,z)\nu_t(dz)dt$
  $\longrightarrow$ **exact** compensation for the *large jumps* of N
- $\sigma_\varepsilon(x) = \int_{t_i}^{t_{i+1}} \int_{-\varepsilon}^{\varepsilon} \sigma^2(x,z)\nu_t(dz)dt$ and (ξ_i) i.i.d $\simeq \mathcal{N}(0,1)$
  $\longrightarrow$ Gaussian **substitute** for *small jumps* of $\tilde{N}$
- $J(x)$ compound Poisson process
  $\longrightarrow$ *large jumps* of N **exactly** simulated

Convergence results for the ε-EM scheme

Theorem 2 (Strong error)**.** *Let $p \ge 2$. Under integrability conditions on b, σ and ν, there exists a version $\widehat{\theta}^\varepsilon$ of the ε-EM scheme (3) such that for $\varepsilon \le n^{-(\frac{1}{2}+\frac{1}{p})}$ we have*

$$\Big\| \sup_{0\le i\le n} |\widehat{\theta}^\varepsilon_{t_i} - \theta_{t_i}| \Big\|_{L^p(\Omega)} \le C n^{-\frac{1}{p}},$$

*where C is a constant that does not depend on n and ε. The rate $n^{-1/p}$ is **optimal** for truncated subordinator Lévy noise.*

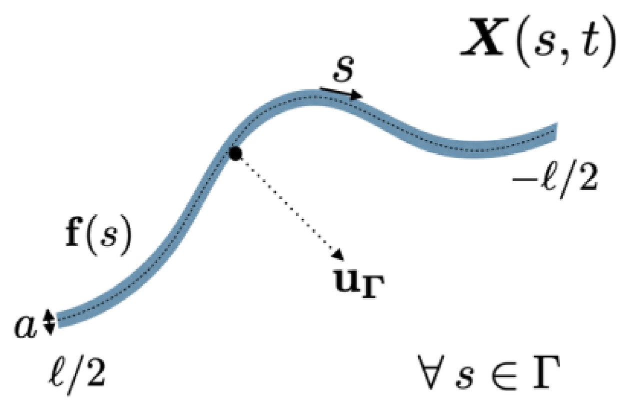
Theorem 3 (Weak error)**.** *Under integrability and regularity conditions on b, σ and ν, for a smooth test function ϕ,*

$$|\mathbb{E}[\phi(\theta_T)] - \mathbb{E}[\phi(\overline{\theta}^\varepsilon_T)]| \le C'(n^{-1} + \varepsilon^{3-\beta}),$$

with Blumenthal-Getoor index $\beta = \inf_{\alpha\in(0,2]} \sup_{t\in[0,T]} \int_{\mathbb{R}} |z|^\alpha \nu_t(dz) < \infty$.

[CBB22] L. Campana, M. Bossy, and J. Bec. *Physical Review Fluids*, 7(12), dec 2022.

Kraichnan S(P)DE model for long rigid fibres



Starting point: slender body SPDE

$$dX_t(s) = \partial u_t(X_t(s)) + (M(X_t)f_t)(s)dt$$

with γ-Hölder singular Kraichnan noise

$$u_t(x) = \sum_{n\in\mathbb{N}} W_t^{(n)} \sigma_\gamma^{(n)}(x)$$

M is a non-linear operator: $M(X) = I_3 + \partial_s X \partial_s X^T$ and f_t stochastic forcing term to ensure the inextensibility constraint $|\partial_s X_t(s)| = 1$.

Rigid fibre assumption $\longrightarrow$ $X_t(s) = \bar{\mathbf{X}}_t + s\, \mathbf{p}(t)$ leads to

$$d\bar{\mathbf{X}}_t = \frac{1}{\ell}\int_{-\ell/2}^{\ell/2} \sigma_\gamma^{(n)}(X_t(s))ds\, \partial W_t^{(n)}$$

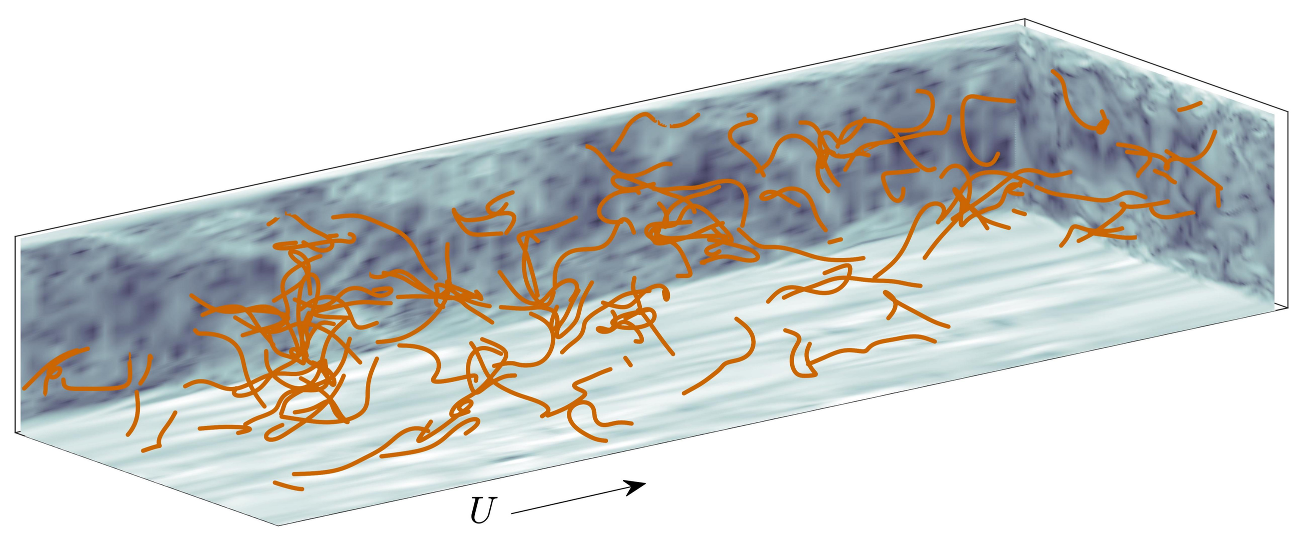
$$d\mathbf{p}(t) = \frac{12}{\ell^3}\int_{-\ell/2}^{\ell/2} s(\sigma_\gamma^{(n)}(X_t(s)) - (\mathbf{p}(t)^T\sigma_\gamma^{(n)}(X_t(s)))\mathbf{p}(t))ds\, \partial W_t^{(n)} \qquad (2)$$

Theorem 1. *For $\gamma \in (0,1)$, there exists a unique strong solution $(\bar{\mathbf{X}}, p)$ in $\mathbb{R}^3 \times \mathcal{S}^1(\mathbb{R}^3)$ to the Stratonovich SDE with infinite-dimensional noise (2) with initial condition $\bar{\mathbf{X}}_0 = 0$ and $p(0) \in \mathcal{S}^1(\mathbb{R}^3)$. Moreover, (2) is equivalent in law to the standard Stratonovich SDE*

$$d\bar{\mathbf{X}}^i_t = \left(c_{(1)}(\ell)\Gamma^s(\mathbf{p}(t))^{ij} + c_{(2)}(\ell)\Gamma^s(\mathbf{p}(t))^{ij}\right)\partial B^j_t$$

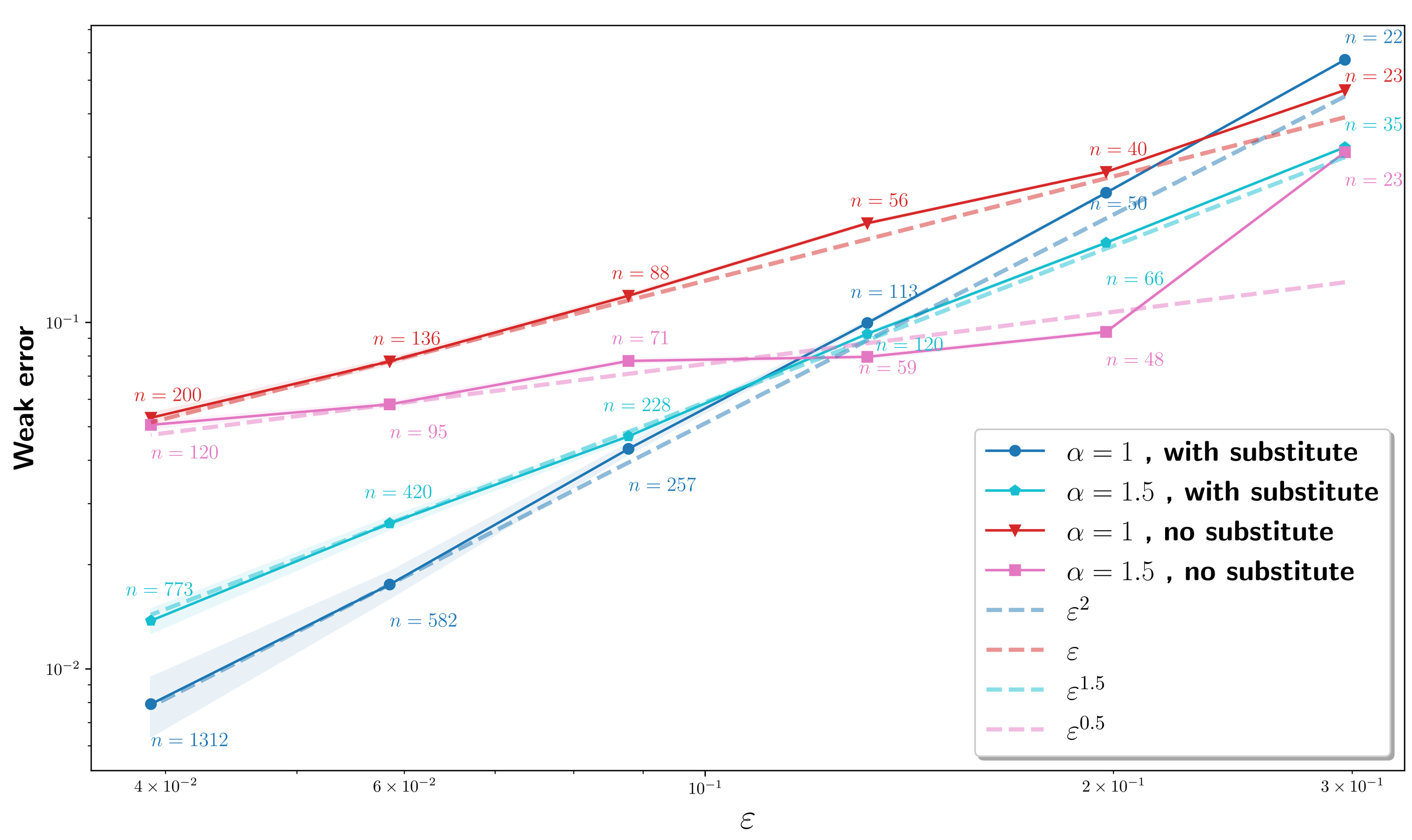
$$d\mathbf{p}(t)^i = c_{(3)}(\ell)\Gamma^a(\mathbf{p}(t))^{ij}\,\partial B^j_t, \quad i = 1,\dots,3 \text{ and } j = 1,\dots,9$$

where $c_{(k)}(\ell) \simeq \ell^{\gamma-1}$, $\Gamma^s(p)$, $\Gamma^a(p) \in \mathbb{R}^{3\times 9}$ and $(B^j)_{1\le j\le 9}$ are i.i.d Brownian motions independent of everything else.



DNS of Navier-Stokes equation, image source from Jérémie Bec.

Numerical simulations



Numerical simulation of the weak error for the ε-EM scheme associated to $d\theta_t = -2\theta_t dt + \sin(\theta_{t-})z\tilde{N}(dt,dz)$