Complex Day 2023

07/02

Stochastic models driven by a Lévy noise

Application to rods orientation in turbulence

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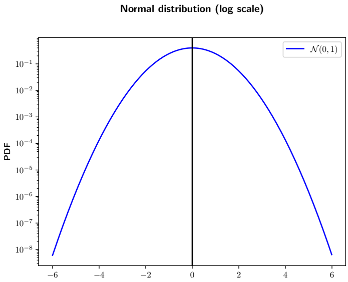
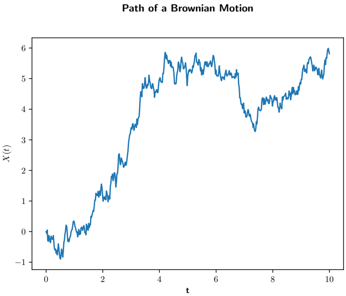
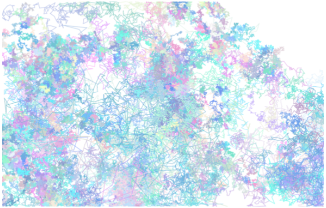
Brownian Motion and normal distribution





Continuous stochastic process $(W_t)_{t\geq 0}$ such that :

- $W_0 = 0$.
- $\forall s, t \in \mathbb{R}_+, W_{t+s} - W_s$ is independent from W_s.
- $\forall s, t \in \mathbb{R}_+, W_{t+s} - W_s \sim \mathcal{N}(0, t)$.



SDEs driven by Brownian Motion

A stochastic process $(X_t)_{t\geq 0}$ is solution of a stochastic differential equation (SDE) if

$$X_t = \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s, \quad (1)$$

where $a, b : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are measurable applications.

Example

In finance, the Black-Scholes formula is obtained by modeling the stock price by the equation

$$X_0 = x_0, \quad X_t = \int_0^t rX_s ds + \int_0^t \sigma X_s dW_s, \quad (2)$$

where r is the interest rate and σ the volatility. This particular SDE can be solved analytically :

$$X_t = x_0 e^{(r - \frac{\sigma}{2})t + \sigma W_t} \quad (3)$$

The option premium is then given by $\mathbb{E}[h(X_T)]$ where h is the payoff and T is the maturity of the option.

- Quantities such as $\mathbb{E}[h(X_T)]$ can be approximated by Monte-Carlo simulation :

$$\mathbb{E}[h(X_T)] \simeq \frac{1}{N}\sum_{i=0}^{N} h(X_T^i) \tag{4}$$

where $(X_T^i)_{i\leq n}$ are independent copies of X_T.

- For this simulation to be possible, we can discretize the process $(X_t)_{t\in[0,T]}$ on $0 = t_0 < ... < t_n = T$, $t_i = i/n$, using the Euler-Maryuama scheme :

$$\bar{X}^n_{t_{i+1}} = \bar{X}^n_{t_i} + a(t_i, \bar{X}^n_{t_i})h + b(t_i, \bar{X}^n_{t_i})\sqrt{h}G, \tag{5}$$

with $h = T/n$ and $G \sim \mathcal{N}(0,1)$. This allows to generate $X_T \simeq \bar{X}^n_{t_n}$

- Results about convergence of this scheme are well known in the litterature :

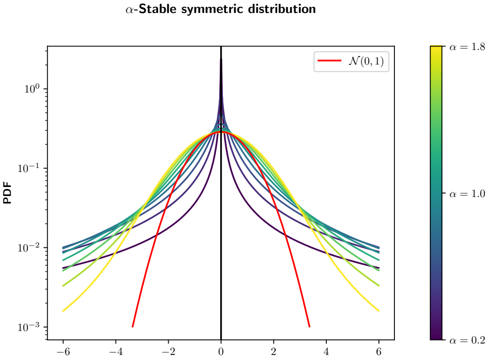
$$\mathbb{E}\left[\sup_{0\leq i\leq n} |X_{t_i} - \bar{X}^n_{t_i}|^2\right] \leq \frac{C_T}{n}, \quad \mathbb{E}[f(X_T) - f(\bar{X}^n_{t_n})] = O\left(\frac{1}{n}\right) \tag{6}$$

Lévy Processes

Càdlàg stochastic process $(L_t)_{t\geq 0}$ such that :

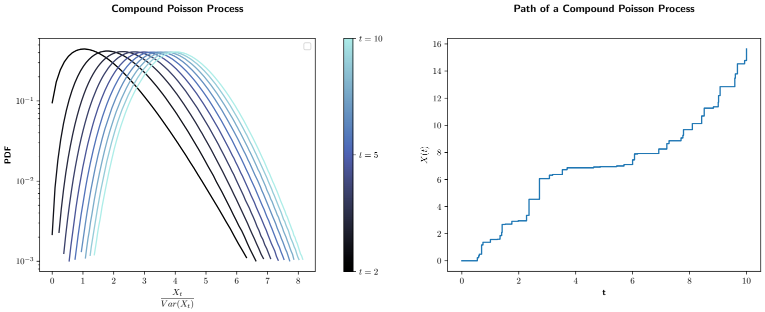
- $\forall s, t \in \mathbb{R}_+$, $L_{t+s} - L_s$ is independent from W_s.
- $\forall s, t \in \mathbb{R}_+$, $L_{t+s} - L_s \sim L_t$.
- Lévy-Kintchine formula : L is characterized in law by the triplet (μ, σ, ν), since

$$\mathbb{E}[e^{ixL_t}] = \exp\left(t\left[i\mu x - \frac{1}{2}\sigma^2 x^2 + \int_{\mathbb{R}^*}(1 - e^{ixy} + ixy1_{\{|y|<1\}})\nu(dy)\right]\right).$$



Example 1 : Compound Poisson Process

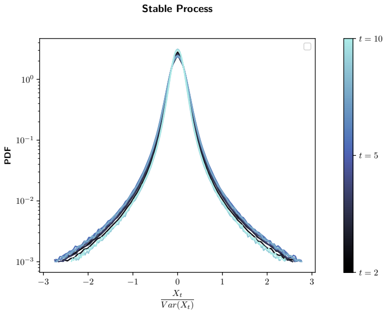
Numerical simulation with $\lambda = 4$ and $Y_1 \sim Exp(3)$

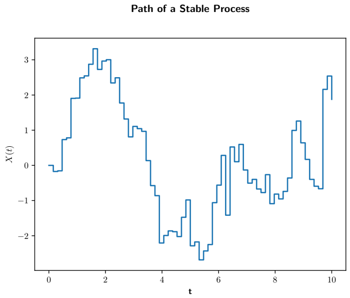


$$X_t = \sum_{i=1}^{N_t} Y_i, \text{ with :}$$

- $(N_t)_{t \in \mathbb{R}_+}$ a Poisson process of intensity λ.
- $(Y_i)_{i \in \mathbb{N}^*}$ i.i.d random variables with distribution π.
- $(\mu, \sigma, \nu) = (0, 0, \lambda\pi)$.

Numerical simulation with $\alpha = 1.5$

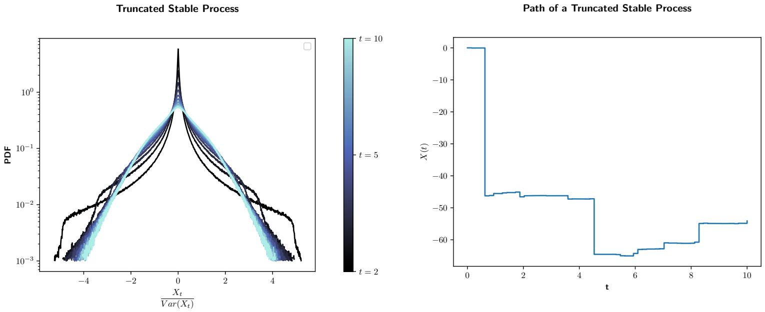




- $(\mu, \sigma, \nu) = (0, 0, \nu_\alpha)$, with $\nu_\alpha(dz) = |z|^{-1-\alpha} dz$, $\alpha \in (0, 2]$.
- $X_{t+h} - X_t \sim h^{1/\alpha} X_1$.
- Easy to generate (Box-Muller like algorithm for X_1) but infinite moments : $\mathbb{E}[|X_1|^\beta] = \infty$ for $\beta \geq \alpha$. In particular, no variance for $\alpha < 2$, hence not convenient to model physical phenomenon.

Example 3 : Truncated Stable Process

Numerical simulation with $\alpha = 0.5$ and $z_* = 100$



- $(\mu, \sigma, \nu) = (0, 0, \nu_\alpha)$, with $\nu_\alpha(dz) = \mathbf{1}_{|z| \le z_*} |z|^{-1-\alpha} dz$.
- $\mathbb{E}[|X_t|^2] = 2t \frac{z_*^{2-\alpha}}{2-\alpha} < \infty$.
- No exact simulation algorithm known for X_1 when $\alpha \in (1, 2]$.

A Lévy process L with triplet (μ, σ, ν) can be written as

$$L_t = \mu t + \sigma W_t + J_t^s + J_t^l, \tag{7}$$

where J^s and J^l designates the "small jumps" and "large jumps" part of L, i.e

$$J_t^l = \int_0^t \int_{|z|>1} zN(ds, dz) \simeq \sum_{i=1}^{N_t^{\lambda(1,\infty)}} Y_i^{1,\infty}.$$

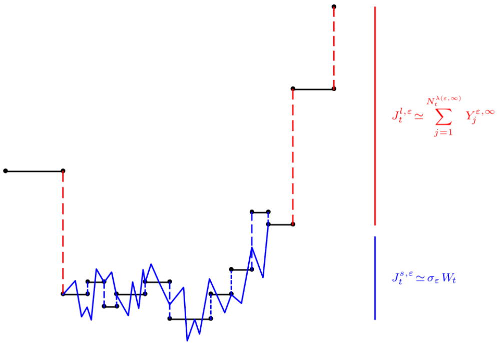
$$J_t^s = \int_0^t \int_{|z|\leq 1} z(N(ds, dz) - \nu(dz)ds) \simeq \lim_{\delta \to 0} \left\{ \sum_{i=1}^{N_t^{\lambda(\delta,1)}} Y_i^{\delta,1} - t \int_{|z|\leq 1} z\nu(dz) \right\}$$

with $\lambda(a, b) = \int_{a\leq |z| \leq b} \nu(dz)$ and $(Y_i^{a,b})_{i\in\mathbb{N}^*}$ i.i.d $\sim \frac{\nu(dz)}{\lambda(a,b)}$.

- The jump measure ν verifies $\int_{-\infty}^{+\infty} min(1, z^2)\nu(dz) < \infty$.

We fix $\varepsilon > 0$ and consider a pure jump Lévy process $L \sim (0, 0, \nu)$.

- The jumps larger than ε corresponds to the compound Poisson process $J_t^{l,\varepsilon} = \sum_{j=1}^{N_t^{\lambda(\varepsilon,\infty)}} Y_j^{\varepsilon,\infty}$, that can be simulated exactly.
- The jumps smaller than ε are approximated by a Brownian motion with the same variance $J_t^{s,\varepsilon} \simeq \sigma_\varepsilon W_t$ where $\sigma_\varepsilon = \sqrt{\mathbb{E}[|J_t^{s,\varepsilon}|^2]}$.
- Then we set $L_t = J_t^{s,\varepsilon} + J_t^{l,\varepsilon}$.



- A stochastic process X is said to solve a SDE driven by a Lévy process if

$$X_t = \int_0^t a(X_s)ds + \int_0^t b(X_s) \int_{-\infty}^{+\infty} \left(N(ds,dz) - \nu(dz)\mathbf{1}_{|z|\leq 1}\right) ds.$$

- For $\varepsilon > 0$, we introduce the scheme $\bar{X}^{n,\varepsilon}$:

$$\bar{X}^{n,\varepsilon}_{t_{i+1}} = \bar{X}^{n,\varepsilon}_{t_i} + a(\bar{X}^{n,\varepsilon}_{t_i}) + \sigma_\varepsilon \sqrt(h)G + b(\bar{X}^{n,\varepsilon}_{t_i}) \sum_{j=N^{\lambda_\varepsilon}_{t_i}+1}^{N^{\lambda_\varepsilon}_{t_{i+1}}} Y^\varepsilon_j,$$

with $\sigma_\varepsilon = \sqrt{\int_{|z|\leq\varepsilon} |z|^2 \nu(dz)}$ and $G \sim \mathcal{N}(0,1)$.

- N. Fournier proved that the following L^2 strong error upper bound holds :

$$\mathbb{E}\left[\sup_{0\leq i\leq n} |X_{t_i} - \bar{X}^{n,\varepsilon}_{t_i}|^2\right] \leq C_T \left(\frac{1}{n} + n(\varepsilon)^2\right),$$

- A stochastic process X is said to solve a SDE driven by a time inhomogeneous Lévy noise if

$$X_t = \int_0^t a(s, X_s)ds + \int_0^t \int_{-\infty}^{+\infty} b(s, X_s, z) \left(N(ds, dz) - \nu_s(dz)\mathbf{1}_{|z|\leq 1} ds\right).$$

- For $\varepsilon > 0$, we introduce the scheme $\bar{X}^{n,\varepsilon}$:

$$\bar{X}^{n,\varepsilon}_{t_{i+1}} = \bar{X}^{n,\varepsilon}_{t_i} + a(t_i, \bar{X}^{n,\varepsilon}_{t_i}) + \sigma_\varepsilon(t_i, \bar{X}^{n,\varepsilon}_{t_i})G + \sum_{j=N^{\lambda_\varepsilon}_{t_i}+1}^{N^{\lambda_\varepsilon}_{t_{i+1}}} b(t_i, \bar{X}^{n,\varepsilon}_{t_i}, Y(T^\varepsilon_j)),$$

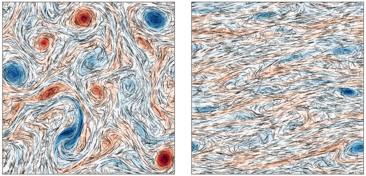
with $\sigma_\varepsilon(\tau, \theta) = \sqrt{\int_{t_i}^{t_{i+1}} \int_{|z|\leq\varepsilon} |b(\tau, \theta, z)|^2 \nu_s(dz)}$, $G \sim \mathcal{N}(0, 1)$,
$T^\varepsilon_j = \inf\{t > 0 : N^{\lambda_\varepsilon}_t = j\}$, $\mathbb{P}(Y(T^\varepsilon_j) \in dx | T^\varepsilon_j = t) = \frac{\nu_t(dz)}{\lambda_\varepsilon}$.

Theorem

If b satisfies $|b(t, x, z)| \leq \bar{b}(\varepsilon)$ for $t \in [0, T]$, $x \in \mathbb{R}$ and $z \in [-\varepsilon, \varepsilon]$, then

$$\mathbb{E}\left[\sup_{0\leq i\leq n} |X_{t_i} - \bar{X}^{n,\varepsilon}_{t_i}|^2\right] \leq C_T \left(\frac{1}{n} + n\bar{b}(\varepsilon)^2\right).$$

Application to the orientation of rods in turbulence



$\sigma^* = 0$ $\sigma^* = 2.8$

Vorticity field ω of the turbulent flow for two different values of the shear σ^*. Blue corresponds to positive values (cyclonic eddies) and red to negative values (anticyclonic). The orientation of the rods are shown as black segments.

Figures provided courtesy of [Campana et al., 2022]

- We consider intertialless rods in a turbulent flow with position equation $dX(t)/dt = v(X(t), t)$, coupled with a unit orientation vector p following Jeffery's equation :

$$\frac{d}{dt}p = \mathbb{A}p - (p^T \mathbb{A}p)p. \tag{8}$$

- After approximations on the gradient tensor $\mathbb{A}$ at the equilibrium regime, the SDE followed by the unfolded angle $\theta_t = \arctan(p_2/p_1)$ is derived

$$\theta_t = \theta_0 + \int_0^t a(\theta_s)ds + \int_0^t b(\theta_s)dW_s, \tag{9}$$

with $a(x)$ and $b(x)$ being linear combining of $\cos(x)$ and $\sin(x)$.

Lorenzo Campana, Mireille Bossy, and Jérémie Bec.
Stochastic model for the alignment and tumbling of rigid fibres in two-dimensional turbulent shear flow, 2022.

The Levy noise model

This Gaussian model however fail to reproduce some of the characteristics present in the direct numerical simulation (DNS).

- The PDF of θ obtained by the DNS shows the presence of heavy tails at small times.
- The process θ also seem to have two regimes, being super-diffusive (i.e $\mathbb{E}[|\theta_t|^2] \sim t^\alpha$ with $\alpha > 1$) at small times, and eventually converging to a diffusive regime (i.e $\mathbb{E}[|\theta_t|^2] \sim t$).

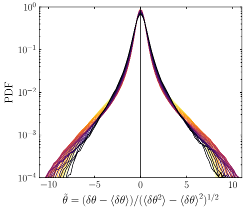
To enhance the diffusive model, we choose to replace the Brownian motion in the SDE by a time inhomogeneous truncated stable process L_t, with Lévy measure

$$\nu_s(dz)ds = \left\{ \sqrt{s}\mathbf{1}_{s<T_*} + \sqrt{T_*}\mathbf{1}_{s\geq T_*} \right\} |z|^{-1-\alpha}\mathbf{1}_{|z|<z_*}. \tag{10}$$

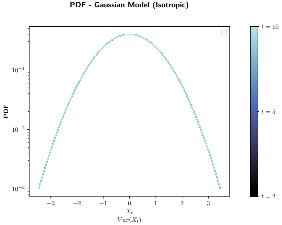
Hence, one can compute

$$\mathbb{E}[|L_t|^2] = \begin{cases} 2\frac{t^{3/2}}{3/2}\frac{z_*^{2-\alpha}}{2-\alpha} & \text{if } t \leq T_* \\ \mathbb{E}[|L_{T_*}|^2] + 2(t-T_*)\frac{z_*^{2-\alpha}}{2-\alpha} & \text{if } t \geq T_*. \end{cases} \tag{11}$$

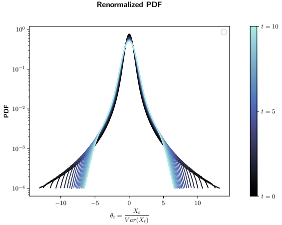
Comparison of the models



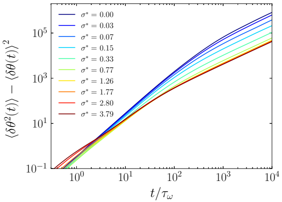
(a). PDF - DNS



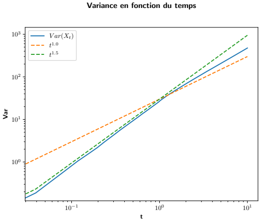
(b). PDF - Gaussian model



(c). PDF - Lévy model

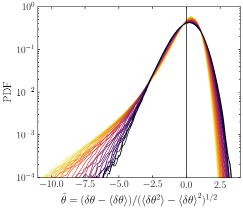


(d). Variance - DNS

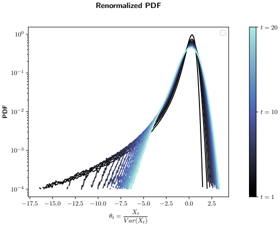


(e). Variance - Lévy Model

- First results in the shear case are promising, though more calibration of the parameters is required.
- In the close future, we plan to extend our results to the multi-dimensional case. As an application, we could build a 3D Lévy noise model for non spherical particles in turbulence. However, the physics of the 3D turbulence is much more complex.
- Another important part of my PhD will be about modelling deformable fibers in turbulence, involving SPDEs analysis, and modelling intermittence with Stochastic Volterra Equations.



(a). PDF - DNS (Shear)



(b). PDF - Lévy Model (Shear, first result)

Thank you for your attention !